Zermelo-Fraenkel Set Theory

H.C. Doets

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Contents

Chapter 1

Introduction

This text comprises an introduction to set theory based on the Zermelo-Fraenkel axiom system ZF (which is probably the most popular axiomatization of the notion of a set), and an introduction to constructibility (and, depending on circumstances, forcing).

The only prerequisite here is some exposure to "naive set theory" (for instance, the appropriate material from [Doets 97]) and some mathematical maturity. Familiarity with formal logic is not presupposed, except in part (B) (Chapters 7 and 8) (but only a little).

For a more elaborate introduction the reader is referred to [Drake/Singh 96], which intends to bridge the gap between easy introductions (a good standard one is [Enderton 77]) and difficult advanced texts [Kunen 80, Jech 78].

Set theory owes its popularity to the fact that it is a unifying system for mathematics: all (or most) of mathematics can be formulated using only the two primitives of set theory: *set* and *element of.* This observation is as important as it is elementary.

Over hundred years ago, Cantor founded his theory of sets on the following basic intuition, the *Comprehension Principle*:

> *Mathematical objects sharing a property form a set, of which they are the members.*

Of course, a set is a particular type of mathematical object; hence it may occur as a member (or *element*) of another set.

Unfortunately, the Comprehension Principle as stated turned out to be somewhat too liberal. Cantor already noticed the fact that it entails "inconsistent" sets; a particularly simple and well-known one is due to Russell: let R be the set of all sets a sharing the property of not occurring among its own elements: $a \notin a$. Then it holds that: $R \in R$ iff $R \notin R$, a (propositional) contradiction.

Nowadays, the usual picture of the set-theoretic universe $\mathbf{V}$ is provided by the *cumulative hierarchy.* $\mathbf{V}$ is approximated by the *partial universes* V_0, V_1, V_2,..., which are built by iterating the notion "subset of", as follows. V_0 is any set of basic objects that are not sets (numbers, lines,...); a next partial universe $\mathrm{V}_{\alpha+1}$ is constructed from the previous one V_α by adding all its subsets ($\wp(A) := \{x \mid x \subset A\}$):

$$\mathrm{V}_{\alpha+1} = \mathrm{V}_\alpha \cup \wp(\mathrm{V}_\alpha).$$

This produces V_n for every natural number index n; however, the sequence of partial universes is much longer than the one of natural numbers; after V_0, V_1, V_2,... follows $\mathrm{V}_\omega = \bigcup_n \mathrm{V}_n$, and you can go on: $\mathrm{V}_{\omega+1} = \mathrm{V}_\omega \cup \wp(\mathrm{V}_\omega)$, etc.

The idea is that, iterating this construction "past every bound one may think of", the universe $\mathbf{V} = \bigcup_\alpha \mathrm{V}_\alpha$ so developed contains all that eventually appears as a member of some such partial universe. In fact, in this construction you can start by taking $\mathrm{V}_0 = \emptyset$, and "lose nothing", as is shown in what follows.

The cumulative hierarchy can be used to motivate the axioms; cf. [Shoenfield 77]. It is the subject of Section 4.5.

Exercises

1 ♣ Assuming that $\mathrm{V}_0 = \emptyset$, compute V_1, V_2 and V_3. How many elements has V_5?

2 ♣ Do not assume that $\mathrm{V}_0 = \emptyset$. Define the sequence $W_0, W_1, W_2, \dots$ by $W_0 = \mathrm{V}_0$ and $W_{n+1} = \mathrm{V}_0 \cup \wp(W_n)$. Show that, for all n, $W_n = \mathrm{V}_n$.

Chapter 2

Axioms

Axioms 0–8 below constitute the *Zermelo Fraenkel* axiom system ZF. Zermelo proved his Well-ordering Theorem in 1904. In 1908, he published a second proof, and a set of axioms, on which his proofs could be based. These axioms are nrs. 1–5 and 7 from the following list, plus the *Axiom of Choice.* Axiom 6 constitutes an addition by Fraenkel and Skolem from 1922 (the system should be called *ZFS* instead of *ZF*). Axiom 8 is due to Mirimanoff and von Neumann.

The axioms below use the two primitives: *set* and *is an element of.* The second one is written '$\in$'.

0. (i) There exists at least one thing. **(ii)** Every thing is a set.

By 0(i), the theory is not trivial. By 0(ii) (more convention than axiom), non-sets will not be discussed. When our axioms are considered to be embedded in a system of logic, 0(i) usually is taken to be a *logical* axiom. It also follows from Axiom 7 below.

1. Extensionality. Sets are completely determined by their elements:

$$\forall a\, \forall b\, [\forall x(x \in a \;\Leftrightarrow\; x \in b) \;\;\Rightarrow\;\; a = b].$$

2. Separation or **Aussonderung.** The elements of a given set (a) sharing a given property (E) form a set:

$$\forall a\, \exists b\, \forall x\, [x \in b \;\;\Leftrightarrow\;\; x \in a \,\wedge\, E(x)].$$

This is Zermelo's weakening of Cantor's Comprehension Principle. Now that Comprehension has been weakened, existence of other sets you want to have must be explicitly postulated. This accounts for Axioms 3–6 below.

Note that, by Extensionality, the set b postulated by the Separation Axiom is uniquely determined by a and E. The usual notation for it is

$$b =_{\text{def}} \{x \in a \mid E(x)\}.$$

If E is a property of sets such that a (by Extensionality, *unique*) set b exists, the elements of which are exactly the sets satisfying E:

$$x \in b \;\;\Leftrightarrow\;\; E(x),$$

then this set b is denoted by $\{x \mid E(x)\}$.
For instance, it holds that $\{x \mid x \in a \,\wedge\, E(x)\} = \{x \in a \mid E(x)\}$ and $\{x \mid x \in b\} = b$.

(Proper) Classes. Constructs of the form $\{x \mid E(x)\}$, where E is a property of sets, are called *classes* or *collections*. In particular, sets are classes. But there may be no set consisting exactly of the objects satisfying E. In that case, the class $\{x \mid E(x)\}$ is called *proper*. Examples of classes that are provably proper are the *universal class* $\mathbf{V} =_{\rm def} \{x \mid x = x\}$ and the Russell class $\{x \mid x \notin x\}$ (cf. Exercises 14 and 15). (In Quine's set theory NF —for: **N**ew **F**oundations— the universal class is a set.)

Note that — by Axiom 0(ii) — for ZF, proper classes simply don't exist, and so the use of abstractions $\{x \mid E(x)\}$ must be regarded as a mere way of speach:

- "$a \in \{x \mid E(x)\}$" is tantamount with "$E(a)$",
- "$\{x \mid E(x)\} = K$" means: "$\forall x(E(x) \Leftrightarrow x \in K)$",
- "$\{x \mid E(x)\} \in K$" means: "$\exists a \in K\ (a = \{x \mid E(x)\})$".

Finally, by "$\{x \mid E(x)\}$ is a set", we mean there is a set a such that $\forall x(x \in a \Leftrightarrow E(x))$. See Exercises 15 and 28 for examples.

The symbol $\mathbf{V}$ is used for the (proper) class $\{x \mid x = x\}$ of all sets. Usually, small letters denote sets; usually, capitals denote classes.

Empty set, Intersection. Take an arbitrary set a (Axiom 0) and define E by: $E(x) \equiv_{\rm def} x \neq x$, then you obtain $\emptyset =_{\rm def} \{x \in a \mid x \neq x\}$, the *empty set*. (Note that $\emptyset = \{x \mid x \neq x\}$ does not depend on the choice of a.) And by using $E(x) \equiv_{\rm def} x \in b$, b any set, you obtain $a \cap b =_{\rm def} \{x \in a \mid x \in b\} = \{x \mid x \in a \wedge x \in b\}$, the *intersection* of a and b.

3. Pairing.

$$\forall a\, \forall b\, \exists c\, \forall x\, (x \in c \Leftrightarrow x = a \vee x = b).$$

That is: $\{x \mid x = a \vee x = b\}$ is a set.

The usual notation for the *unordered pair* of a and b postulated by this axiom is: $c =_{\rm def} \{a, b\}$. N.B.: $\{a, b\} = \{b, a\}$. In case $a = b$, we obtain $\{a\} =_{\rm def} \{a, a\}$, the *singleton* of a.

4. Sumset.

$$\forall a\, \exists b\, \forall x\, [x \in b \Leftrightarrow \exists y(x \in y \wedge y \in a)].$$

That is: $\{x \mid \exists y(x \in y \wedge y \in a)\}$ is a set.

Definition 2.1

1. The set $\bigcup a = \bigcup_{y \in a} y =_{\rm def} \{x \mid \exists y \in a(x \in y)\}$, the existence of which is postulated by the sumset axiom, is called the *sumset* of a.
2. $a \cup b =_{\rm def} \bigcup\{a, b\}$ is called the *union* of a and b.
3. $\{a_0, \ldots, a_n\} =_{\rm def} \{a_0\} \cup \cdots \cup \{a_n\}$. □

5. Powerset.

$$\forall a\, \exists b\, \forall x\, (x \in b \Leftrightarrow x \subset a),$$

that is: $\{x \mid x \subset a\}$ is a set.
Here, $\subset$ is defined by: $x \subset a \equiv_{\rm def} \forall y(y \in x \Rightarrow y \in a)$.

The notation for the set b postulated —the *powerset* of a— is: $b = \wp(a)$.

6. Substitution or **Replacement.** If F is an operation that associates sets with the elements of a set a, then there is a set

$$F[a] =_{\text{def}} \{F(x) \mid x \in a\} = \{y \mid \exists x \in a(y = F(x))\}$$

the elements of which are all sets of the form $F(x)$ where x is an element of a.

Examples of operations that you have encountered sofar are $\{\,.\,\}$ and $\wp$. Hence, if a is a set, then, by the Substitution Axiom, also $\{\{x\} \mid x \in a\}$ and $\{\wp(x) \mid x \in a\}$ are sets.

7. Infinity. There exists a set a for which $\emptyset \in a$ and $\forall x \in a(x \cup \{x\} \in a)$.

This formulation of the infinity axiom uses some definitions which, in turn, are based on other axioms. Therefore a better formulation would be: there exists a set a that contains an elementless set and such that $\forall x \in a\, \exists y \in a\, \forall z(z \in y \;\Leftrightarrow\; z \in x \vee z = x)$.

Note that a set a as postulated by the infinity axiom contains at least the elements $\emptyset$, $\{\emptyset\} = \emptyset \cup \{\emptyset\}$, $\{\emptyset, \{\emptyset\}\} = \{\emptyset\} \cup \{\{\emptyset\}\}$, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, ... Clearly, these are all *different* objects (they contain resp. 0, 1, 2, 3,... elements!: for $\emptyset$ and $\{\emptyset\}$, this is obvious; *hence* these sets are different, *hence* $\{\emptyset, \{\emptyset\}\}$ has two elements, *hence* it differs from the first two etc. etc.) and so intuitively this axiom implies the existence of an infinite set. (N.B.: Existence of *infinitely many sets* already follows from Axioms 0, 1, 2 and 3.)

8. Regularity or **Foundation.**

$$\forall a\, [a \neq \emptyset \;\Rightarrow\; \exists x \in a(x \cap a = \emptyset)].$$

Definition 2.2 We say that the relation $\prec$ is *well-founded* on A if every non-empty subset X of A has a *$\prec$-minimal* element, that is: an element $y \in X$ such that for no $z \in X$ we have that $z \prec y$.

Thus, the Foundation Axiom says that $\in$ is well-founded on the universe of all sets.

However, since the meaning of this statement will be far from clear yet, the axiom will be avoided for the time being, and exercises should be solved without it, unless this is specifically indicated. (For some applications, cf. Exercise 11.)

Later on, the *Axiom of Choice* follows; with that addition, the system is denoted by ZFC.

Language of set theory; Skolem interpretation. The presentation of the axioms above makes evident that — with the exception of two — they can be considered first-order sentences in a language with only one non-logical symbol: the binary relation symbol $\in$. This is the *language of set theory.* The exceptions are the Axioms of Separation and of Substitution. The first one involves the notion of a *property*, the second one that of an *operation.* However, it is not completely clear what is meant by these notions. As a way out of this predicament, Skolem in 1922 proposed that one should admit only properties (in the Separation Axiom) and operations (in the Substitution Axiom) that are *expressible in the first-order language of set theory.* Remarkably, instances of these axioms not so expressible are never needed (for the purposes of the usual theorems). From now on, this *Skolem-interpretation* of the ZF-axioms is followed. Note that, viewing things this way, there are *infinitely many* Separation- and Substitution-Axioms. *Separation Axioms* are all formulas of the form

$$\forall a\, \exists b\, \forall x\, (x \in b \;\Leftrightarrow\; x \in a \wedge \Phi)$$

where Φ is a first-order formula in the language of set theory (that is not allowed to contain free occurrences of the variable b, but that usually contains x free and often several other variables as well). *Substitution Axioms* are all formulas of the form

$$\forall x \in a\, \exists! y\, \Phi \;\Rightarrow\; \exists b\, \forall y\, (y \in b \;\Leftrightarrow\; \exists x \in a\, \Phi).$$

The notation $\exists!\Phi(y)$ is shorthand for: *there exists exactly one y such that* $\Phi(y)$. It is first-order expressible as, e.g., $\exists y \Phi(y) \wedge \forall y \forall z(\Phi(y) \wedge \Phi(z) \Rightarrow y = z)$. The premiss $\forall x \in a\, \exists! y\, \Phi$ expresses that Φ defines an operation on the elements of a. (Again, b is not allowed to occur freely in Φ.)

Definition 2.3 The following definitions present the usual set-theoretic simulations of the notions of ordered pair, relation and function.

1. $(x, y) =_{\mathrm{def}} \{\{x\}, \{x, y\}\}$ is called the *ordered pair* of x and y.
   (Cf. Exercise 8 for the justification of this.)
2. For $n \geqslant 2$, n-tuples can be defined by $(x_0, \ldots, x_n) =_{\mathrm{def}} ((x_0, \ldots, x_{n-1}), x_n)$, as well as by $\{\{x_0\}, \{x_0, x_1\}, \ldots, \{x_0, \ldots, x_n\}\}$ (the *ordered n-tuple* of $x_0, \ldots, x_n$).
3. A *relation* is a set of ordered pairs.
4. A *function* is a relation f with the property that: $(x, y), (x, z) \in f \Rightarrow y = z$.
   (The usual notations in the context of functions are employed.) □

Exercises

3 ♣ Check that axioms 0, 1, 3, 4, 5 and 6 are all valid in the one-element model $(\{0\}, \{(0, 0)\})$.

4 ♣ Try to verify that the axioms are satisfied in the universe of sets given by the cumulative hierarchy.

5 ♣ Deduce the Pairing Axiom from the other axioms.
Hint. First, show that a set exists that has at least two elements.

6 ♣ Deduce the Separation Axiom from the other axioms.
Hint. Use $\emptyset$. (Existence of $\emptyset$ follows from Infinity.)

7 ♣ Show that for any sets A and B, the Cartesian product

$$A \times B =_{\mathrm{def}} \{(a, b) \mid a \in A \wedge b \in B\} \qquad (= \{c \mid \exists a \in A \exists b \in B c = (a, b)\})$$

exists as a set.

Hint. The standard proof uses the Separation Axiom on the set $\wp(\wp(A \cup B))$; a more elegant one uses the Substitution Axiom (twice!) and the operation G on A defined by: $G(a) =_{\mathrm{def}} \{a\} \times B$.

8 ♣ Show:

1. $\{p, q\} = \{p, r\} \;\Rightarrow\; q = r$,

2. $(x, y) = (a, b) \ \Rightarrow \ x = a \wedge y = b$.

9 ♣ Show that if $\{\{x\}, \{\emptyset, y\}\} = \{\{a\}, \{\emptyset, b\}\}$, then $x = a$ and $y = b$. Thus, defining $(x, y) =_{\text{def}} \{\{x\}, \{\emptyset, y\}\}$ would also be a good ordered pair.

10 ♣ Show: if $a \neq \emptyset$, then $\bigcap a =_{\text{def}} \{x \mid \forall y \in a(x \in y)\}$ exists as a set. This even holds when a is a proper class.
What about $\bigcap \emptyset$?

11 ♣ Show, using the Foundation Axiom:

1. No set a is an element of itself. (Hint: consider $\{a\}$.)
2. There are no sets a_1, a_2, a_3, a_4 such that $a_1 \in a_2 \in a_3 \in a_4 \in a_1$.

12 ♣ Suppose that $\forall x \in a\ \exists! y\ \Phi$ holds. Show that $\{(x, y) \mid x \in a \wedge \Phi\}$ is a set (and, hence, a function).

(Hence: if H is an operation on the set a, a function f on a is defined by putting, for $x \in a$: $f(x) =_{\text{def}} H(x)$.)

13 ♣ Why do you think the variable b is not allowed to occur free in Φ in Separation- and Substitution Axiom?

14 ♣ Suppose that the operation F maps the class K injectively into the class L. Assume that K is a proper class. Show that L also is proper.

In particular, if $K \subset L$ and K is proper, then so is L. Example: $\mathcal{R} = \{x \mid x \notin x\} \subset \{x \mid x = x\} = \mathbf{V}$.

15 ♣ Show that the following classes are proper:

- $\mathcal{R} =_{\text{def}} \{x \mid x \notin x\}$, the Russell class of sets that do not belong to themselves,
- $\mathbf{V} = \{x \mid x = x\}$, the class of all sets,
- $\{\{x, y\} \mid x \neq y\}$, the class of all two-element sets,
- $\mathcal{Q}_n =_{\text{def}} \{x \mid \neg \exists x_1, \ldots, x_n (x_1 \in x \wedge x_2 \in x_1 \wedge \cdots \wedge x_n \in x_{n-1} \wedge x \in x_n)\}$, Quine's classes,
- $\mathbf{G} =_{\text{def}} \{x \mid \forall a\, [x \in a \ \Rightarrow \ \exists y \in a(y \cap a = \emptyset)]\}$, the class of *grounded* sets.

Note that $\mathbf{G} \subset \mathcal{Q}_n \subset \mathcal{R} \subset \mathbf{V}$.

N.B.: The Axiom of Foundation says precisely that $\mathbf{V} = \mathbf{G}$. Indeed,

$$\mathbf{V} = \mathbf{G}, \text{ or: } \forall b (b \in \mathbf{G})$$

means that

$$\forall b \forall a\, (b \in a \ \Rightarrow \ \exists x \in a(x \cap a = \emptyset)),$$

which amounts to

$$\forall a\, (\exists b (b \in a) \ \Rightarrow \ \exists x \in a(x \cap a = \emptyset)),$$

which is the Foundation Axiom.

16 ♣ Show:

1. if $x \subset \mathbf{G}$, then $x \in \mathbf{G}$; i.e.: $\wp(\mathbf{G}) \subset \mathbf{G}$,
2. if $x \in \mathbf{G}$ and $y \in x$, then $y \in \mathbf{G}$; i.e.: $\mathbf{G} \subset \wp(\mathbf{G})$.

Hints. 1. Suppose that $x \subset \mathbf{G}$. To show $x \in \mathbf{G}$, assume $x \in a$; in order to find $y \in a$ such that $y \cap a = \emptyset$, distinguish $x \cap a = \emptyset$ and $x \cap a \neq \emptyset$.
2. Suppose $x \in \mathbf{G}$ and $y \in x$. To show $y \in \mathbf{G}$, suppose $y \in a$ and consider $a \cup \{x\}$.

17 ♣ Let A be a set. Show that the following subsets of A do not belong to A. In particular, there is no set A for which $\wp(A) \subset A$.

- $\mathcal{R}_A =_{\text{def}} \{x \in A \mid x \notin x\}$,
- $\mathcal{Q}_{n,A} =_{\text{def}} \{x \in A \mid \neg\exists x_1, \ldots, x_n(x_1 \in x \ \wedge\ x_2 \in x_1 \ \wedge\ \cdots \ \wedge\ x_n \in x_{n-1} \ \wedge\ x \in x_n)\}$,
- $\mathbf{G}_A =_{\text{def}} \{x \in A \mid \forall a(x \in a \ \Rightarrow\ \exists y \in a \forall z \in a(z \notin y))\}$.

Note that to show $\mathcal{R}_A \notin A$ and $\mathcal{Q}_{n,A} \notin A$ you won't need any ZF axiom; to show that $\mathbf{G}_A \notin A$ you will need that one-element classes are sets (a consequence of Pairing), alternatively, that one-element subclasses of A are sets (a consequence of Separation).
In a definite way, these results are consequences of those of Exercise 15 (but note that you cannot prove $A \notin A$ —corresponding to $\mathbf{V} \notin \mathbf{V}$— without Foundation). Can you explain this?

Chapter 3

Natural Numbers

3.1 Peano Axioms

In order to see that ZF is strong enough to develop all (or most) of mathematics, the notions and objects from mathematics have to be defined —better: simulated— in ZF. The main mathematical notion is that of a function, and this one could be introduced thanks to the possibility of a set-theoretic definition of *ordered pair* (Definition 2.3.1 p. 8). As regards the mathematical objects: here follows the (set of) natural numbers; from these, the other number systems (rationals, reals) may be defined in the well-known way.

The set of natural numbers can be characterized ("up to isomorphism" — see Theorem 3.1) using the *Peano Axioms*. More precisely, these axioms characterize the system $(\mathbb{N}, 0, \mathrm{S})$, where S is the *successor-operation* on $\mathbb{N}$ defined by: $\mathrm{S}(n) =_{\mathrm{def}} n+1$. The Peano Axioms are the following five statements about this system:

1. 0 is a natural number: $0 \in \mathbb{N}$,
2. the successor of a natural number is a natural number: $n \in \mathbb{N} \Rightarrow \mathrm{S}(n) \in \mathbb{N}$,
3. S is injective: $\mathrm{S}(n) = \mathrm{S}(m) \Rightarrow n = m$,
4. 0 is not a successor: for all $n \in \mathbb{N}$, it holds that $0 \neq \mathrm{S}(n)$,
5. *("mathematical") induction:*
   if $X \subset \mathbb{N}$ is such that (i) $0 \in X$ and (ii) $\forall n \in X(\mathrm{S}(n) \in X)$, then $\mathbb{N} \subset X$.

Let us call a *Peano system* any system (A, a_0, s) that satisfies the Peano axioms.

An *isomorphism* between such systems (A, a_0, s) and (B, b_0, t) is a bijection $h : A \to B$ for which $h(a_0) = b_0$ and such that for all $a \in A$, $h(s(a)) = t(h(a))$, and systems between which such an isomorphism exists are called *isomorphic*. The idea is that isomorphic systems are complete lookalikes.

Theorem 3.1 *Every two Peano systems are isomorphic.*

Proof. Sketch. (But see Exercise 31 p. 17.)

Assume that (A, a_0, s) and (B, b_0, t) are Peano systems. Define $a_n =_{\mathrm{def}} s(\cdots s(a_0)\cdots)$. ($n$ occurrences of 's').
Claim. a_n is different from $a_0, \ldots, a_{n-1}$.
Proof. Induction w.r.t. n.

Basis. $n = 0$. There is nothing to prove.
Induction step. Assume that a_n is different from $a_0, \ldots, a_{n-1}$ (induction hypothesis). By the fourth axiom, $a_{n+1} \neq a_0$. And if $a_{n+1} = a_{m+1}$, $0 \leqslant m < n$, then by the third axiom $a_n = a_m$, contradicting the induction hypothesis.
Claim. $A = \{a_n \mid n \in \mathbb{N}\}$.
Proof. If not, $X =_{\text{def}} \{a_n \mid n \in \mathbb{N}\}$ would not satisfy the induction postulate.
Of course, similar things are true of the system (B, b_0, t) and the sequence of b_n. The required isomorphism maps a_n to b_n $(n \in \mathbb{N})$.

Note that the theorem cannot (yet) be considered a formal result of ZF as $\mathbb{N}$ has not (yet) been defined. □

3.2 Set-theoretic Definition of $\mathbb{N}$

The above theorem shows that just *any* Peano system can be used to simulate $(\mathbb{N}, 0, \mathrm{S})$. A particularly simple one is given by the following choice. Since it is meant to simulate $(\mathbb{N}, 0, \mathrm{S})$, the symbols 0 and S are used to describe it.

1. $0 =_{\text{def}} \emptyset$
2. $\mathrm{S}(x) =_{\text{def}} x \cup \{x\}$.

These stipulations provide us with definitions for all individual natural numbers:
$1 = \mathrm{S}(0) = 0 \cup \{0\} = \{\emptyset\} = \{0\}$,
$2 = \mathrm{S}(1) = 1 \cup \{1\} = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$,
$3 = \mathrm{S}(2) = \cdots = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\}$,
...
The main attraction of this particular way of defining integers is, that we'll have $n = \{m \in \mathbb{N} \mid m < n\}$ —see later.

Intuitively, it is now clear what the *set of natural numbers* will be. However, a *formal definition* of this set (or at least of the *concept* of a natural number) is still lacking, and we need such a definition for Theorem 3.3. If you've never seen it before, the following definition may look just a trick.

Definition 3.2

1. A set a is $(0, \mathrm{S}$-) *closed* if $0 \in a$ and $\forall x \in a(\mathrm{S}(x) \in a)$.
2. A *natural number* is something that is an element of every closed set. □

At this point, you may like to check that 0, 1, 2,... are natuarl numbers in the sense of this definition (see Theorem 3.5).

Theorem 3.3 *The natural numbers form a set.*

Proof. The Infinity Axiom says that closed sets exist. Let a be one of them. Then by the Separation Axiom

$$\{x \in a \mid x \text{ is a natural number}\}$$

is a set; and it is clear that its elements are exactly the natural numbers. □

Notation 3.4 ω denotes the set of natural numbers. □

Theorem 3.5 $(\omega, 0, \mathrm{S})$ *is a Peano system.*

Proof.
1. By definition, 0 is in every closed set, and hence it belongs to their intersection ω.
2. Assume that $n \in \omega$. That is: n is in every closed set. Then $\mathrm{S}(n)$ is in every closed set (if a is closed, then $n \in a$; hence $\mathrm{S}(n) \in a$). Therefore, $\mathrm{S}(n) \in \omega$.
4. 0 is not a successor: 0 is empty, and a successor never is empty ($x \in \mathrm{S}(x)$).
5. (Induction.) Assume that the set $X \subset \omega$ is such that (i) $0 \in X$ and (ii) $\forall n \in X(\mathrm{S}(n) \in X)$. Then X is closed. Therefore, $\omega \subset X$. (If $n \in \omega$, then n is in every closed set; in particular, it is in X.)
It remains to verify the third Peano axiom. This needs a couple of lemmas: see below. □

Lemma 3.6 *If* $i \in j$*,* $j \in n$ *and* $n \in \omega$*, then* $i \in n$*.*

Proof. Induction w.r.t. n.
I.e., put $X =_{\text{def}} \{n \in \omega \mid \forall j \in n \forall i \in j (i \in n)\}$ and show that X is closed.
Basis. That $0 \in X$ is trivial.
Induction step. Assume the *induction hypothesis*, that $n \in X$.
To show that $\mathrm{S}(n) \in X$, assume that $i \in j \in \mathrm{S}(n) = n \cup \{n\}$. Then $j \in n$ or $j \in \{n\}$, i.e.: $j = n$. Now if $j \in n$, then $i \in n$ follows from the induction hypothesis. And if $j = n$, then $i \in n$ is clear. In both cases, $i \in \mathrm{S}(n)$. □

Lemma 3.7 *If* $n \in \omega$*, then* $n \notin n$*.*

Proof. Induction w.r.t. n.
Basis: for $n = \emptyset$, the result is trivial.
Induction step. Assume as an induction hypothesis that $n \notin n$. Suppose now that $\mathrm{S}(n) \in \mathrm{S}(n) = n \cup \{n\}$. (i) $\mathrm{S}(n) \in n$. Then by Lemma 3.6, $\mathrm{S}(n) \subset n$, and $n \in n$. (ii) $\mathrm{S}(n) = n$. Then $n \in n$ follows as well. □

Finally, the third Peano axiom.

Lemma 3.8 *If* $n, m \in \omega$ *and* $\mathrm{S}(n) = \mathrm{S}(m)$*, then* $n = m$*.*

Proof. Assume that $\mathrm{S}(n) = \mathrm{S}(m)$, i.e., that $n \cup \{n\} = m \cup \{m\}$. Since $n \in n \cup \{n\}$, it follows that $n \in m \cup \{m\}$ as well. Thus, $n \in m$ or $n \in \{m\}$. Now $n \in \{m\}$ amounts to: $n = m$, which was to be proved. Therefore, assume that $n \in m$.
Exchanging the roles of n and m in this argument, similarly obtain that $m \in n$. By Lemma 3.6, it follows that $n \in n$, contradicting Lemma 3.7. □

Definition 3.9 A set or class A is called *transitive* if for all $y \in A$, if $x \in y$, then $x \in A$. □

By Lemma 3.6, every natural number is a transitive set: its proof shows that (i) $\emptyset$ is transitive, and (ii) that if a set n is transitive, then so is its successor $S(n)$.

It is probably safe to say that the only transitive set that occurs in "ordinary" mathematics (as opposed to set theory) is the empty set. Sets that occur outside the domain of pure set theory are usually "typed" in a rigid way that prevents transitivity: the elements of a set are either non-sets (for instance, numbers, points, functions, …), or sets of non-sets, or sets of … of non-sets.

Exercises

18 ♣ Show that a set A is transitive iff $\bigcup A \subset A$, iff $A \subset \wp(A)$.

19 ♣ Suppose that A is a non-empty transitive set. Assume the Foundation Axiom. Show that $\emptyset \in A$.

20 ♣ Show that ω is transitive.

21 ♣ Show that $\omega \notin \omega$.

22 ♣ Are the following (always) true? Prove, or provide a counter-example.

1. If every $x \in A$ is transitive, then so is $\bigcup A$.
2. If A is transitive, then so is $\bigcup A$.
3. If $\bigcup A$ is transitive, then so is A.
4. If A is transitive, then so is $\wp(A)$.
5. If $\wp(A)$ is transitive, then so is A.

23 ♣ Show that the classes $\mathcal{G} =_{\text{def}} \{x \mid \forall a(x \in a \Rightarrow \exists y \in a(y \cap a = \emptyset))\}$ and $\mathcal{Z} =_{\text{def}} \{x \mid \neg\exists f : \omega \to \mathbf{V}[f(0) = x \wedge \forall n(f(n+1) \in f(n))]\}$ from Exercise 15 resp., 28 are transitive.

3.3 Order, Induction and Recursion

Note that for every (concrete) natural number n it holds that $n = \{0, \dots, n-1\}$. Therefore the following simple definition is intuitively correct.

Definition 3.10 For $n, m \in \omega$: $n < m \equiv_{\text{def}} n \in m$. □

However, this definition can also be formally justified. First, note that the following two principles certainly should be provable, since they hold for the ("true") ordering of the ("true") natural numbers.

Lemma 3.11 *1.* $m \not< 0$,

2. $m < \mathrm{S}(n) \Leftrightarrow m < n \vee m = n$.

Proof. Completely trivial. □

But now, any relation $\prec$ on ω that satisfies these principles must coincide with $<$:

Lemma 3.12 *Suppose that the relation $\prec$ on ω is such that*

1. $m \not\prec 0$,

2. $m \prec \mathrm{S}(n) \Leftrightarrow m \prec n \vee m = n$.

Then $\prec$ coincides with $<$.

Proof. Using induction w.r.t. n, we show that

$$m < n \Rightarrow m \prec n.$$

The converse implication follows by symmetry.

First, if $n = 0$, then (by 1) neither $m < n$ nor $m \prec n$ hold. Next, assume $m < \mathrm{S}(n)$. By 2 (for $<$), $m < n$ or $m = n$. By IH, $m \prec n$ or $m = n$. By 2 (for $\prec$), $m \prec \mathrm{S}(n)$. □

Thus, the principles of Lemma 3.11 actually *characterize* the ordering, which implies that, since $<$ satisfies them, it must be the true ordering.

It follows that one —at least, in principle— never has to refer back to Definition 3.10: the two properties from Lemma 3.11 always suffice.

Theorem 3.13 (Strong Induction) $\forall n[\forall m < n\, \Phi(m) \Rightarrow \Phi(n)] \Rightarrow \forall n \Phi(n)$.

Proof. Define $X =_{\text{def}} \{n \in \omega \mid \forall m < n\, \Phi(m)\}$. Assume that $\forall n[\forall m < n\, \Phi(m) \Rightarrow \Phi(n)]$. That is: $n \in X \Rightarrow \Phi(n)$. Therefore, for $\forall n \Phi(n)$ to hold, it suffices to show, that $X = \omega$. This is accomplished by ordinary induction.
Basis. That $0 \in X$ is trivial (by Lemma 3.11.1).
Induction step. Assume that $n \in X$. Then by definition of X, $\forall m < n\, \Phi(m)$, and by assumption it follows that $\Phi(n)$. Therefore (by Lemma 3.11.2), $\forall m < \mathrm{S}(n)\, \Phi(m)$; i.e., $\mathrm{S}(n) \in X$. □

Theorem 3.14 $<$ *is a linear ordering of* ω.

Proof.
Irreflexive (for no $n \in \omega$, $n < n$): this is Lemma 3.7.
Transitive (for all $i, j, n \in \omega$, if both $i < j$ and $j < n$, then $i < n$): this is Lemma 3.6.
Linear: i.e., for all $n, m \in \omega$: $m < n \vee n < m \vee n = m$. Use strong induction *twice.* Use Exercise 24. Alternatively, use ordinary mathematical induction twice, first showing that $\forall m \in \mathbb{N}(0 \leqslant m)$. (This part will be generalized in Theorem 4.6 p. 25; the proof given there can be copied for this case.) □

Definition 3.15
1. Assume $n \in \omega$. A set A *has* n *elements* or *has cardinality* n, notation: $|A| = n$, if there exists a bijection between A and n.
2. A is *finite* if for some $n \in \omega$, A has cardinality n.
3. A set that is not finite is called *infinite.*

□

As a familiar example of recursion, note that addition $+$ on ω is completely described by the following two equations:

1. $n + 0 = n$
2. $n + \mathrm{S}(m) = \mathrm{S}(n + m)$.

These *recursion equations* "define" addition "by recursion". Given the *parameter* n, the first equation tells us what it means to add 0; the second one explains what it is to add $\mathrm{S}(m)$ in terms of addition of m. (For instance, to compute $3 + 2$ using these equations, you obtain: $3 + 2 = 3 + \mathrm{S}(1) = \mathrm{S}(3 + 1) = \mathrm{S}(3 + \mathrm{S}(0)) = \mathrm{S}(\mathrm{S}(3 + 0)) = \mathrm{S}(\mathrm{S}(3)) = 5$.) Analogous equations for multiplication and exponentiation are possible.

The general form of such recursion equations is the following:

1. $F(n,0) = G(n)$
2. $F(n, \mathrm{S}(m)) = H(n, m, F(n,m))$.

Theorem 3.16 (Recursion on $\mathbb{N}$) *For any two operations G ($G : \omega \to \mathbf{V}$) and H ($H : \omega \times \omega \times \mathbf{V} \to \mathbf{V}$) there is exactly one operation F ($F : \omega \times \omega \to \mathbf{V}$) satisfying the above equations.*

Note that, if you compute $F(n,m)$ using these equations, this boils down to computing the sequence of values $F(n,0), F(n,1), F(n,2), \ldots, F(n,m)$, that form an *approximation* of the operation F. The following lemma states in effect existence and uniqueness of such approximations.

Lemma 3.17 *For all $n, m \in \omega$ there exists exactly one function f such that* $\mathrm{Dom}(f) = \mathrm{S}(m)$ *and*

1. $f(0) = G(n)$

2. if $k < m$, then $f(\mathrm{S}(k)) = H(n, k, f(k))$.

Proof.
At most one: If f and f' both satisfy the conditions, then it follows by induction w.r.t. k that $k \leqslant m \;\Rightarrow\; f'(k) = f(k)$. Hence, $f' = f$.
At least one: Induction w.r.t. m.
Basis. For $m = 0$, take $f =_{\text{def}} \{(0, G(0))\}$.
Induction step. Assume that f satisfies the conditions w.r.t. n and m.
Then $f \cup \{(\mathrm{S}(m), H(n, m, f(m)))\}$ satisfies the conditions w.r.t. n and $\mathrm{S}(m)$. □

Proof of Theorem 3.16.
Define $F(n,m) =_{\text{def}} f(m)$, where f is the unique function defined on $\mathrm{S}(m)$ that is given by Lemma 3.17. □

The parameter n from the Recursion Theorem 3.16 does not need to be a natural number: it can be an arbitrary set (or a finite sequence of sets). For that case, the proofs given remain valid; however, you cannot longer conclude that F is a *function* since its domain is not a set: F now is an *operation*. The next section starts with an example of such a recursion.

Exercises

24 ♣ Show that, for $n, m \in \omega$:

1. $n \leqslant m \;\Leftrightarrow\; n \subset m$,
2. $n < m \;\Leftrightarrow\; \mathrm{S}(n) \leqslant m$.

25 ♣ Use Strong Induction to show (Lemma 3.7) that for all $n \in \omega$, $n \notin n$.

26 ♣ Suppose that $\emptyset \neq A \subset \omega$ and $\bigcup A = A$. Show that $A = \omega$.

27 ♣ Show: $\exists n \in \omega \Psi(n) \;\Rightarrow\; \exists n \in \omega[\Psi(n) \wedge \forall m < n \neg\Psi(m)]$.
Hint. Apply strong induction w.r.t. n to show that, instead, for all $n \in \omega$, $\Psi(n) \;\Rightarrow\; \exists n \in \omega[\Psi(n) \wedge \forall m < n \neg\Psi(m)]$.

28 ♣ Define the property Z by: $Z(x) \equiv_{\rm def}$ there is no function f defined on ω such that (i) $f(0) = x$ and (ii) for all $n \in \omega$: $f(n+1) \in f(n)$. Show that the class $\mathcal{Z} =_{\rm def} \{x \mid Z(x)\}$ is not a set, and that for every set A: $\{x \in A \mid Z(x)\} \notin A$.

29 ♣ Give recursion equations for multiplication and exponentiation.

30 ♣ Check that the operation F defined in the proof of Theorem 3.16 indeed satisfies the recursion equations.

31 ♣ Show that Theorem 3.1 is, in fact, a theorem of ZF.

32 ♣ Show that mathematical induction is equivalent with the statement that for every set X: $\forall n[0 \in X \wedge \forall m < n(m \in X \Rightarrow \mathrm{S}(m) \in X) \Rightarrow n \in X]$.

(Note that this makes sense also for *finite* sets X — in contrast with the usual formulation of induction.)

33 ♣ Give an adequate definition of *natural number* that is not based on the Infinity Axiom. (So there may not exist a *set* of natural numbers.)
Hint. Transform the principle from Exercise 32 into a suitable definition.

34 ♣ Construct some $0, \mathrm{S}$-closed sets different from ω.
Hint. Show that *every* set is subset of a closed set.

35 ♣ Show:

1. a subset of a finite set is finite,
2. a finite union of finite sets is finite,
3. a product of finite sets is finite,
4. the powerset of a finite set is finite,
5. if a and b are finite, then there are only finitely many functions from a to b,
6. if a and b both have n elements and $a \subset b$, then $a = b$,
7. if a is finite and $f : a \to a$, then f is injective iff it is surjective,
8. ω is inifinite,
9. if a has n elements and $n \neq m$, then a doesn't have m elements.

3.4 Transitive Closure

Definition 3.18

1. Recursively, define the binary operation TC on $\mathbf{V} \times \omega$ by
    - $\mathrm{TC}(a, 0) = a$
    - $\mathrm{TC}(a, \mathrm{S}(m)) = \bigcup \mathrm{TC}(a, m)$.

2. The *transitive closure* $\mathrm{TC}(a)$ of the set a is defined by
   $\mathrm{TC}(a) =_{\mathrm{def}} \bigcup_{m\in\omega} \mathrm{TC}(a,m)\ (= \bigcup\{\mathrm{TC}(a,m) \mid m \in \omega\})$. □

Parts 1, 2 and 3 of the next lemma say that $\mathrm{TC}(a)$ is the least transitive set containing a.

Lemma 3.19

1. $a \subset \mathrm{TC}(a)$,
2. $\mathrm{TC}(a)$ *is transitive,*
3. *if* $b \supset a$ *is transitive, then* $\mathrm{TC}(a) \subset b$,
4. $\mathrm{TC}(a) = a \cup \bigcup_{b\in a} \mathrm{TC}(b)$.

Definition 3.20 Let R be a binary relation. Define $R^\star$, the *transitive closure* of R, by:
$a\, R^\star\, b :\equiv_{\mathrm{def}}$
$\exists n{\in}\omega \exists f[\mathrm{Dom}(f) = n+2 \ \wedge\ f(0) = a \ \wedge\ f(n+1) = b \ \wedge\ \forall i < n+1(f(i)Rf(i+1))]$. □

Alternatively, you might recursively define $R_0 = R$ and $R_{n+1} = R_n \circ R$ (where $S \circ R =_{\mathrm{def}} \{(x,z) \mid \exists y(xSy \ \wedge\ yRz)\}$); then $R^\star =_{\mathrm{def}} \bigcup_n R_n$. However, this does not work in case R is a binary property that is not a set.
Example. The transitive closure of the successor relation on ω (the relation defined by $\mathrm{S}(x) = y$) is the ordering relation $<$.

Lemma 3.21 (Transitive Closure)

1. $R \subset R^\star$,
2. $R^\star$ *is transitive (considered as a relation:* $a\, R^\star\, b \ \wedge\ b\, R^\star\, c \ \Rightarrow\ a\, R^\star\, c$*),*
3. *if* $R \subset S$ *and* S *is transitive, then* $R^\star \subset S$,
4. $a\, R^\star\, c \Leftrightarrow aRc \ \vee\ \exists b[a\, R^\star\, b \ \wedge\ bRc]$.

Exercises

36 ♣ Prove Lemma 3.19.1–3. Prove Lemma 3.19.4, and do *not* use 3.18, but use 3.19.1–3.

37 ♣ Parts 1–3 of 3.19 characterize the operation TC: Assume that the operation TC′ satisfies the properties expressed by Lemma 3.19.1/2/3. Show that for all a, $\mathrm{TC}'(a) = \mathrm{TC}(a)$.
Something similar holds w.r.t. $R^\star$ and Lemma 3.21.1/2/3. Formulate and prove this.

38 ♣ Prove Lemma 3.21.

39 ♣ Show that $x \in \mathrm{TC}(a)$ iff $x \in^\star a$.

40 ♣ A relation R is *confluent* if $\forall a \forall b \forall c(aRb \ \wedge\ aRc \ \Rightarrow\ \exists d(bRd \ \wedge\ cRd))$. Show that if R is confluent, then so is $R^\star$.

3.5 Inductive Definitions

Chances are, that the set-theoretic definition of ω (Definition 3.2, p.12) as the intersection of all closed sets does not correspond very well to your intuition of a natural number as generated from 0 by S in a (finite) number of steps. Definition 3.2 is the simplest non-trivial example of a so-called *inductive definition*. Theorem 3.24 abstracts some important features from the natural number context.

Definition 3.22 Let A be a class or a set, and let the operator H map subclasses of A to subclasses of A (if A is a set, this simply means: $H : \wp(A) \to \wp(A)$).

1. H is called *monotone* if $X \subset Y \subset A \Rightarrow H(X) \subset H(Y)$.
2. $K \subset A$ is called
   (a) *H-closed* or a *pre-fixed point* of H if $H(K) \subset K$,
   (b) a *fixed point* of H if $H(K) = K$,
   (c) *inductive* if for every $X \subset A$ such that $H(X) \subset X$, we have that $K \subset X$,
   (d) *least fixed point (lfp)* if it is a fixed point that is included in every fixed point.

Running Example. Let A be an $0, \mathrm{S}$-closed set (i.e.: $0 \in A$ and $\forall a \in A(\mathrm{S}(a) \in A)$). Define H on $\wp(A)$ by: $H(X) =_{\text{def}} \{0\} \cup \{\mathrm{S}(a) \mid a \in X\}$.

Note that pre-fixed points and $0, \mathrm{S}$-closed sets are the same. Since A is closed, you have: $H : \wp(A) \to \wp(A)$. Clearly, H is monotone.

Lemma 3.23 1. *Every operator has at most one inductive pre-fixed point.*

2. *The inductive pre-fixed point of a monotone operator is a least fixed point.*

Proof. 1. Trivial: if K is a pre-fixed point and K' is inductive, then $K' \subset K$.

2. Assume that K is the inductive pre-fixed point of a monotone operator H. Thus, $H(K) \subset K$. By monotonicity, $H(H(K)) \subset H(K)$. By inductivity, $K \subset H(K)$. Thus, $H(K) = K$. □

Running example, continued. Note that $\omega = \bigcap\{X \in \wp(A) \mid H(X) \subset X\}$, the intersection of all pre-fixed points of H, is the inductive fixed point of H. This observation in the context of the definition of ω is now generalized by the following theorem.

Theorem 3.24 (The Least Fixed Point) *Let A be a set. If $H : \wp(A) \to \wp(A)$ is monotone, then H has a least fixed point.*

Proof. Let $F =_{\text{def}} \{X \in \wp(A) \mid H(X) \subset X\}$ be the set of all pre-fixed points of H. Note that $F \neq \emptyset$, since $A \in F$. So, the class $I =_{\text{def}} \bigcap F$ is, in fact, a subset of A. Now:
1. $X \in F \Rightarrow I \subset X$. (Obvious.)
2. $I \in F$.
Proof: Assume that $X \in F$, i.e., that $H(X) \subset X$. By 1, $I \subset X$. By monotonicity, $H(I) \subset H(X)$. Therefore, $H(I) \subset X$.
Since X was an arbitrary element of F, it holds that $H(I) \subset \bigcap F = I$: I is a pre-fixed point of H.
By 1 and 2, I is the least pre-fixed point of H.
3. $I \subset H(I)$: By 1, 2 and Lemma 3.23.2. □

Note: In this proof, it is essential that A is a set, or at least: that some pre-fixed point is a set: cf. the definition of I as an intersection of such sets. There are several situations where we'd like to use an inductive definition over a proper class but have to use some ad-hoc solution. E.g., $\mathbf{G}$ is the least fixed point of the *powerclass*-operation

$$\wp : X \mapsto \{y \in \mathbf{V} \mid y \subset X\}$$

(Theorem 4.18, p. 32), and OR is the least fixed point of

$$\mathbf{T} : X \mapsto \{y \in \mathbf{V} \mid y \subset X \text{ is transitive}\}$$

(Theorem 4.3, p. 24).
Functions, operations, operators. A function is a set of ordered pairs satisfying a certain uniqueness condition. An operation F must be given by a formula $\Phi(x, y)$ for which $\forall x \exists! y \Phi(x, y)$ is true. In that case, '$F(x) = y$' is tantamount with '$\Phi(x, y)$'. An operation maps sets to sets. By an *operator*, we often mean an association of (proper) *classes* to *classes*. Two examples are displayed above. Such an operator Γ is always given by means of a formula $\Phi(X, y)$, where X is a free variable for classes. Then $\Gamma(X) = \{y \in \mathbf{V} \mid \Phi(X, y)\}$. It is not the intention to formally introduce variables for classes here: note that each time a particular definition $\Psi(x)$ of a class X is given (i.e., that $X = \{x \in \mathbf{V} \mid \Psi(x)\}$), then $\Gamma(X)$ can be calculated by (i) obtaining $\Phi'(y)$ by replacing expressions '$x \in X$' by '$\Psi(x)$' in $\Phi(X, y)$, thereby eliminating every occurrence of X, and (ii) letting $\Gamma(X) = \{y \in \mathbf{V} \mid \Phi'(y)\}$.

Definition 3.25 The least fixed point of a monotone operator is said to be *inductively defined* by it.
The least fixed point of H is denoted by $H{\uparrow}$. □

Running example, cont'd. For $H(X) = \{0\} \cup \{\mathrm{S}(x) \mid x \in X\}$, Peano's Induction Axiom amounts to:

$$H(X) \subset X \;\Rightarrow\; \omega \subset X$$

which was immediate from the definition of ω. The abstract version of this is part 1 of the above proof.
Part 2 of the proof is the abstract version of the fact that ω is closed.
Finally, part 3 of the proof abstracts the fact that every natural number is 0 or is the successor of a natural number. (Show this using induction.)

In the context of the Fixed Point Theorem 3.24, by *induction* w.r.t. $I =_{\text{def}} \bigcap\{X \mid H(X) \subset X\}$ (an application of) the implication $H(X) \subset X \;\Rightarrow\; I \subset X$ is meant.

Definition 3.26 The operator $H : \wp(A) \to \wp(A)$ is *finite* (*finitary, compact*) if for all X and a: if $a \in H(X)$, then a finite $Y \subset X$ exists such that $a \in H(Y)$. □

The following theorem gives a more constructive approach to the least fixed point of a finite operator.

Theorem 3.27 *Assume that $H : \wp(A) \to \wp(A)$ is monotone and finite. Recursively, define $H{\uparrow} : \omega \to \wp(A)$ by*

- $H{\uparrow}0 = \emptyset$,

- $H{\Uparrow}\mathrm{S}(n) = H(H{\Uparrow}n)$.

(The sets $H{\Uparrow}n$ are called stages *in the least fixed point construction.)*
Then the set $H{\Uparrow}\omega =_{def} \bigcup_{n\in\omega} H{\Uparrow}n$ is the least fixed point of H.

Running example, cont'd. For $0, \mathrm{S}$-closed A, $H(X) =_{\mathrm{def}} \{0\} \cup \{\mathrm{S}(x) \mid x \in X\}$, and $H{\Uparrow}n$ as defined in Theorem 3.27 it holds that $H{\Uparrow}n = n$.

More on the approximation of fixed points in Section 4.4 (p. 30).

Exercises

41 ♣ Assume that A is a set and $H : \wp(A) \to \wp(A)$ is monotone. Show: H has a *greatest* fixed point, denoted by $H{\downarrow}$.
Hint. A *post-fixed point* is a set X for which $X \subset H(X)$. The greatest fixed point simultaneously is the greatest post-fixed point.

There does not appear to be a result similar to Theorem 3.27 pertaining to greatest fixed points. There are finite monotone operators over a set A for which ω-fold iteration starting from A does not result in the greatest fixed point. (But see the results in Section 4.4.)

42 ♣ Assume that $G, H : \wp(A) \to \wp(A)$ are monotone operators such that for all $X \subset A$, $G(X) \subset H(X)$. Suppose that $G{\uparrow}$ and $H{\uparrow}$ are the least fixed points of G resp. H, and that $G{\downarrow}$ and $H{\downarrow}$ are the respective greatest fixed points. Show that $G{\uparrow} \subset H{\uparrow}$ and $G{\downarrow} \subset H{\downarrow}$.

43 ♣ $\mathbb{Z}$ is the set of integers. Define $H : \wp(\mathbb{Z}) \to \wp(\mathbb{Z})$ by $H(X) =_{\mathrm{def}} \{0\} \cup \{\mathrm{S}(x) \mid x \in X\}$. Identify the fixed points of H.

44 ♣ Prove Theorem 3.27.
Hint. Do *not* use Theorem 3.24. Show that $n < m \;\Rightarrow\; H{\Uparrow}n \subset H{\Uparrow}m$. Show: if $H(X) \subset X$, then, for all n, $H{\Uparrow}n \subset X$. Finally, show that $H(H{\Uparrow}\omega) \subset H{\Uparrow}\omega$. (For this, you will need the fact that if Y is finite and $Y \subset \bigcup_{n\in\omega} H{\Uparrow}n$, then for some $m \in \omega$, $Y \subset H{\Uparrow}m$. This is shown by induction w.r.t. the *number of elements* of Y, cf. Definition 3.15, p.15.)

45 ♣ (This shows that finiteness is needed for Theorem 3.27.) Let $A = \omega \cup \{\omega\}$ and define $H : \wp(A) \to \wp(A)$ by $H(X) = \{0\} \cup \{\mathrm{S}(x) \mid x \in X\}$ if $\omega \not\subset X$, and $H(X) = A$ otherwise. Show: H is monotone, H is not finite, $H{\uparrow} = A$, $\forall n \in \omega\; H{\Uparrow}n = n$. Thus, $H{\uparrow} \neq \bigcup_n H{\Uparrow}n$.

46 ♣ Assume that R is a relation on the set A.

1. Show that the transitive closure $R^\star$ of R (cf. Definition 3.20) is the least fixed point of the operation $H : \wp(A^2) \to \wp(A^2)$ defined by $H(X) =_{\mathrm{def}} R \cup \{(x, z) \mid \exists y[(x, y) \in X \wedge yRz]\}$.
2. Show that $R^\star$ also is least fixed point of the operation $H'(X) =_{\mathrm{def}} R \cup \{(x, z) \mid \exists y[(x, y) \in X \wedge (y, z) \in X]\}$.

Hint for 1. Lemma 3.21.4 says that it is a fixed point.

47 ♣ Let $H : \wp(B) \to \wp(B)$ be monotone with least fixed point I and assume $I \subset A \subset B$. Define $H_A : \wp(A) \to \wp(A)$ by $H_A(X) =_{\mathrm{def}} A \cap H(X)$. Show: I is least fixed point of H_A as well.

48 ♣ Let $H : \wp(B) \to \wp(B)$ be monotone with least fixed point I and assume $B \subset C$. Define $H^C : \wp(C) \to \wp(C)$ by $H^C(X) =_{\text{def}} H(B \cap X)$. Show: I is least fixed point of H^C as well.

49 ♣ Prove Theorem 3.1 by inductively defining the required isomorphism.

50 ♣ Inductively define the function F from Theorem 3.16, p.16.

51 ♣ (Simultaneous inductive definitions.) Suppose that $\Pi, \Delta : \wp(A) \times \wp(A) \to \wp(A)$ are monotone operators in the sense that if $X_1, Y_1, X_2, Y_2 \subset A$ are such that $X_1 \subset X_2$ and $Y_1 \subset Y_2$, then $\Pi(X_1, Y_1) \subset \Pi(X_2, Y_2)$ (and similarly for Δ). Show that K, L exist such that

1. $\Pi(K, L) \subset K$, $\Delta(K, L) \subset L$; in fact, $\Pi(K, L) = K$, $\Delta(K, L) = L$,
2. if $\Pi(X, Y) \subset X$ and $\Delta(X, Y) \subset Y$, then $K \subset X$ and $L \subset Y$.

Show that, similarly, *greatest* (post-) fixed points exist. Generalize to more operators.

52 ♣ Suppose that $H : \wp(A) \to \wp(A)$ is monotone. The *dual* of H is the operator $H^d : \wp(A) \to \wp(A)$ defined by $H^d(X) =_{\text{def}} A - H(A - X)$. H^d is monotone. Relate its (least, resp., greatest) fixed points to those of H.

53 ♣ ♣ Suppose that $H : \wp(A) \times \wp(A) \to \wp(A)$ is monotone in *both* arguments. For $X \subset A$, define $H_X : \wp(A) \to \wp(A)$ by $H_X(Y) = H(X, Y)$. Define $J : \wp(A) \to \wp(A)$ by $J(X) = H_X\!\uparrow$. Define $I : \wp(A) \to \wp(A)$ by $I(X) = H(X, X)$. Show that $J\!\uparrow = I\!\uparrow$.

Chapter 4

Ordinals

Natural numbers have (at least) *two* uses: to count the position of an element in a finite ordering (*ordinal* use), and to count the number of elements in a finite set (*cardinal* use). Since the order in which the elements of a finite set are counted doesn't influence the final outcome, the distinction between these two roles goes usually unnoticed. However, the situation changes completely when it comes to infinite sets. (For instance, there are many –in fact, uncountably many— countably infinite well-order types.) Ordinal numbers are introduced in the present chapter; cardinal numbers are the topic of Chapter 6.

4.1 Definition

In classical set theory, an *ordinal* is the order type of a well-ordering. Von Neumann discovered particularly simple objects —the *Von Neumann ordinals*— that may be taken as substitutes. The natural numbers as introduced above are examples of Von Neumann-ordinals. The natural numbers are the objects you get by starting with 0 and applying the successor S "any number of times".

Note that every number n equals the set $\{0, \ldots, n-1\}$ of its predecessors. So, instead of generating the naturals using 0 and S, you can obtain them also by the process

$$0, \ldots, n-1 \mapsto n$$

that generates a transitive object ($\{0, \ldots, n-1\}$) from its elements ($0, \ldots, n-1$). Since 0 has no predecessors, it is generated for free; subsequently, all numbers 1, 2, 3 ... are generated; but now, the process does not stop after generating all of ω, but it goes on, generating
$\omega = \{0, 1, 2, \ldots\}$, $\mathrm{S}(\omega) = \omega + 1 = \{0, 1, 2, \ldots, \omega\}$, $\omega + 2 = \{0, 1, 2, \ldots, \omega, \omega + 1\}$, $\omega + 3, \ldots;$
$\omega + \omega = \omega \cdot 2 = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots\}$, $\omega \cdot 2 + 1$,..., $\omega \cdot 2 + \omega = \omega \cdot 3, \ldots;$
$\omega \cdot \omega = \omega^2, \ldots$,$\omega^3, \ldots$ $\omega^\omega, \ldots$ $\omega^{\omega^\omega}, \ldots$

Intuitively, what an ordinal is, is now clear. However, just as in the case of the natural numbers, a formal *definition* of the concept of an ordinal is still lacking. In the case of ω, the solution consisted in the Axiom of Infinity — which provides one example of a $0, \mathrm{S}$-closed set — coupled with an inductive definition — the operator $H(X) =_{\mathrm{def}} \{0\} \cup \{\mathrm{S}(x) \mid x \in X\}$ — producing the least $0, \mathrm{S}$-closed set ω. In the case of the ordinals, there is a problem.

What the inductive definition should look like is rather clear: the monotone operator to employ clearly is

$$\mathbf{T}(X) =_{\text{def}} \{a \mid a \subset X \ \wedge \ a \text{ is transitive } \}.$$

Examples: $\mathbf{T}(\emptyset) = \{\emptyset\}$, $\mathbf{T}(\{0, \dots, n-1\}) = \{0, \dots, n\}$, $\mathbf{T}(\{0, 1, 2 \dots\}) = \{0, 1, 2 \dots, \omega\}$.

What we're looking for is a class OR for which the following two principles hold:

Closure: $\mathbf{T}(\text{OR}) \subset \text{OR}$;

Induction: *For all* X, *if* $\mathbf{T}(X) \subset X$, *then* $\text{OR} \subset X$.

Note that these principles can be satisfied by at most one class OR: see Lemma 3.23.1 (p. 19).

Just as was the case with defining ω, we would be able to identify OR as a set if we could find at least one set Ω that satisfies Closure (then OR would be the smallest one). Note that Theorem 3.24 (p. 19) requires the universe A over which the inductive definition is carried out, to be a *set*. However:

Proposition 4.1 *There is no set* Ω *such that* $\mathbf{T}(\Omega) \subset \Omega$.

Proof. It suffices to identify, for an arbitrary set A, a transitive subset $B \subset A$ such that $B \notin A$. The following is an example of such a subset:

$$\{x \in A \mid \text{TC}(x) \subset A\} \cap \mathbf{G},$$

where $\mathbf{G} = \{x \mid \forall a(x \in a \ \Rightarrow \ \exists y \in a(y \cap a = \emptyset))\}$ is the class of "grounded" sets (see Exercise 15 p. 9).

Note that both $\{x \in A \mid \text{TC}(x) \subset A\}$ and $\mathbf{G}$ are transitive (see Exercise 16 p. 10), and hence so is their intersection.

Next, note that $\text{TC}(B) \subset A$ (for, $\text{TC}(B) = B \subset A$).

Finally, $B \in \mathbf{G}$: for, if $B \in a$, then either $B \cap a = \emptyset$ (so a is disjoint from one of its elements), or $B \cap a \neq \emptyset$, say, $x \in B \cap a$, but then $x \in \mathbf{G}$ and a is disjoint from one of its elements in this case as well.

If, moreover, $B \in A$ holds, then it follows that $B \in B$. But then $\{B\}$ wouldn't be disjoint with one of its members. □

So, if there is a class OR of ordinal numbers, this certainly cannot be a set. (This observation is due, be it in a somewhat different context, to Burali-Forti.) We cannot use Theorem 3.24 to identify OR.

The following definition has the required properties.

First, let $\mathbf{TR}$ be the class of transitive sets, and put $\mathbf{TRR} =_{\text{def}} \{x \in \mathbf{TR} \mid x \subset \mathbf{TR}\}$.

Again: $\mathbf{G} = \{x \mid \forall a(x \in a \ \Rightarrow \ \exists y \in a(y \cap a = \emptyset))\}$.

Definition 4.2 $\text{OR} =_{\text{def}} \mathbf{TRR} \cap \mathbf{G}$.

Theorem 4.3

1. $\mathbf{T}(\text{OR}) \subset \text{OR}$ *(Closure),*
2. $\mathbf{T}(X) \subset X \ \Rightarrow \ \text{OR} \subset X$ *(Induction).*

Proof. 1. Closure. Assume $a \in \mathbf{T}(\mathrm{OR})$. I.e.: $a \in \mathbf{TR}$, and $a \subset \mathrm{OR} = \mathbf{TRR} \cap \mathbf{G}$. Then $a \in \mathbf{TRR}$, $a \in \mathbf{G}$ (see Exercise 16 p. 10), and $a \in \mathrm{OR}$.

2. Induction. Aiming for a contradiction, assume that $\mathbf{T}(X) \subset X$, $\alpha \in \mathrm{OR}$, and $\alpha \notin X$.

Claim: $\alpha \in \mathbf{T}(X)$ (and contradiction).

Proof: (i) $\alpha \in \mathbf{TR}$: obvious.

(ii) $\alpha \subset X$: for if not, then some $y \in \alpha - X$ exists. Then $y \in \mathbf{G}$ (again, see Exercise 16 p. 10). Hence, $y \in \alpha - X$ exists s.t. $y \cap (\alpha - X) = \emptyset$.

Claim: $y \in \mathbf{T}(X)$ (and contradiction).

Proof: That $y \in \mathbf{TR}$ is obvious. Also, $y \subset X$: for if $z \in y$, then $z \in \alpha$, hence $z \in X$. □

From now on, small greek letters usually denote ordinals.

Definition 4.4 $\alpha < \beta \equiv_{\text{def}} \alpha \in \beta$. □

Induction for OR, that is: Theorem 4.3.2, is usually presented in the following guise.

Theorem 4.5 *"Transfinite Induction":*
If $K \subset \mathrm{OR}$ is such that $\forall\alpha \in \mathrm{OR}(\forall\beta < \alpha(\beta \in K) \Rightarrow \alpha \in K)$, then $\mathrm{OR} \subset K$.

Proof. Assume that $\forall\alpha(\forall\beta < \alpha(\beta \in K) \Rightarrow \alpha \in K)$.

Claim: $\mathbf{T}(\mathrm{OR} \cap K) \subset \mathrm{OR} \cap K$.

From this, by Induction, we get that $\mathrm{OR} \subset \mathrm{OR} \cap K$, and the result follows.

Proof of Claim: Suppose that $a \in \mathbf{T}(\mathrm{OR} \cap K)$. Since $\mathrm{OR} \cap K \subset \mathrm{OR}$, we have that $\mathbf{T}(\mathrm{OR} \cap K) \subset \mathbf{T}(\mathrm{OR}) = \mathrm{OR}$; hence $a \in \mathrm{OR}$. In order that $a \in K$, it suffices (by assumption on K) to show that $\forall\beta < a(\beta \in K)$, that is: $a \subset K$. However, this is immediate from $a \in \mathbf{T}(\mathrm{OR} \cap K)$. □

Theorem 4.6 $<$ *linearly orders*[1] OR.

Proof. Irreflexivity is immediate from transfinite induction. Transitivity is trivial. To show

$$\forall\alpha\forall\beta(\alpha < \beta \vee \beta < \alpha \vee \alpha = \beta),$$

we use Transfinite Induction. Thus, let α be an arbitrary ordinal, and assume as a (first) induction hypothesis that

$$\forall\alpha' < \alpha\forall\beta(\alpha' < \beta \vee \beta < \alpha' \vee \alpha' = \beta).$$

We have to show now, that

$$\forall\beta(\alpha < \beta \vee \beta < \alpha \vee \alpha = \beta).$$

Again, we apply Transfinite Induction. This time, let β be an arbitrary ordinal, and assume as a second induction hypothesis that

$$\forall\beta' < \beta(\alpha < \beta' \vee \beta' < \alpha \vee \alpha = \beta').$$

[1] A relation $\prec$ is a *linear ordering* of a class A if it is irreflexive, transitive, and for all $a, b \in A$: if $a \neq b$, then either $a \prec b$ or $b \prec a$ holds.

We have to show now that

$$\alpha < \beta \ \vee \ \beta < \alpha \ \vee \ \alpha = \beta.$$

Assume, moreover, that $\alpha \not< \beta$ and $\beta \not< \alpha$. Now

$$\forall \alpha' < \alpha(\alpha' < \beta), \text{ and } \forall \beta' < \beta(\beta' < \alpha)$$

easily follow from the two IH's and the fact that ordinals are transitive, and this entails $\alpha = \beta$ by the Extensionality Axiom. □

Definition 4.7 α is a *successor* if, for some β, $\alpha = \mathrm{S}(\beta)$. Instead of $\mathrm{S}(\beta)$, one usually writes $\beta + 1$. A non-zero ordinal that is not a successor is called a *limit*. □

The smallest limit is ω. Existence of other limits needs the Substitution Axiom. Note that α is a limit iff $\alpha \neq 0$ and α has no greatest element.

Exercises

54 ♣ Assume that the set a is transitive. Show:

1. $a \in \mathbf{G}$ iff $\in$ is well-founded on a,
2. $a \subset \mathbf{TR}$ iff $\in$ is transitive on a.

Thus, an ordinal is the same as a transitive set on which $\in$ is a transitive and well-founded relation. (This is the standard definition of the notion.)

55 ♣ Show that **TRR** is the greatest fixed point of **T**.

56 ♣ Show:

1. $0 \in \mathrm{OR}$,
2. $\alpha \in \mathrm{OR} \ \Rightarrow \ \mathrm{S}(\alpha) \in \mathrm{OR}$,
3. $\omega \subset \mathrm{OR}$,
4. $\omega \in \mathrm{OR}$.

57 ♣ Show that $\{\{\emptyset\}\} \notin \mathrm{OR}$. Show that $\wp(\wp(\wp(\emptyset))) \notin \mathrm{OR}$.

58 ♣ Show:

1. $\alpha \leqslant \beta \ \Leftrightarrow \ \alpha \subset \beta$,
2. if K is a non-empty class of ordinals, then $\bigcap K$ is the least element of K ($\bigcap K \in K$ and $\forall \alpha \in K(\bigcap K \leqslant \alpha)$),
3. if K is a *set* of ordinals, then $\bigcup K$ is an ordinal that is the *sup* of K (that is: the least ordinal $\geqslant$ every $\alpha \in A$),
4. if K is a proper class of ordinals, then $\bigcup K = \mathrm{OR}$.

59 ♣ Assume that $K \subset \mathrm{OR}$ is such that

- $0 \in K$,
- $\alpha \in K \;\Rightarrow\; \alpha + 1 \in K$,
- if γ is a limit and $\forall \xi \in \gamma\, (\xi \in K)$, then $\gamma \in K$.

Show: every ordinal is in K.
Hint. Induction.

60 ♣ We know that for every set A (i) there is a set $B \subset A$ such that $B \notin A$, and (ii) there is a *transitive* set $B \subset A$ such that $B \notin A$.
Show: (iii) there is an *ordinal* $\beta \subset A$ such that $\beta \notin A$. Can you prove this without using the Substitution Axiom?

4.2 Well-order Types

A well-ordering is a linear ordering that is well-founded. Classically, an ordinal is a *well-order type.* Theorem 4.10 shows that you can view ordinals as such.

Lemma 4.8 *Suppose that $(A, \prec)$ is a well-ordering. For every order-preserving function $h : A \to A$ it holds that $\forall a \in A(a \preceq h(a))$.*
In particular, $(A, \prec)$ cannot be order-preservingly mapped into a proper initial.

Proof. Suppose that $h : A \to A$ is order-preserving ($a \prec b \;\Rightarrow\; h(a) \prec h(b)$) but for some $a \in A$ we have that $h(a) \prec a$. Using that $\prec$ well-orders A, let $b \in A$ be the *least* element such that $h(b) \prec b$. Then also $h(h(b)) \prec h(b)$. These properties of $b' = h(b)$ contradict minimality of b. □

Corollary 4.9 *If $(\alpha, <) \cong (\beta, <)$ then $\alpha = \beta$.*

Proof. If $\beta < \alpha$, then β is a proper initial of $(\alpha, <)$. Apply Lemma 4.8. □

Theorem 4.10 *1. Every structure $(\alpha, \in)$ (where $\alpha \in \mathrm{OR}$) is a well-ordering,*

2. for every well-ordering $(A, \prec)$ (where A is a set) there is exactly one ordinal α such that $(A, \prec) \cong (\alpha, <)$.

Proof. 1. If $\emptyset \neq x \subset \alpha$, say, $\beta \in x$, then $\beta \in \mathbf{G}$ and hence x has a least element.
2. At most one: Immediate from the previous Corollary.
At least one: For the purposes of this proof, let us call an injection f *good* if $\mathrm{Dom}(f)$ is an initial of A (that is: if $a \prec b$, and $b \in \mathrm{Dom}(f)$, then $a \in \mathrm{Dom}(f)$), $\mathrm{Ran}(f) \in \mathrm{OR}$, and for all $a, b \in \mathrm{Dom}(f)$: $a \prec b \;\Leftrightarrow\; f(a) < f(b)$.
(i) If f and g are good, and $a \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$, then $f(a) = g(a)$.
This is (almost) immediate from Corollary 4.9: the composition $g \circ f^{-1}$ is an order-isomorphism between the ordinals $f(a)$ and $g(a)$.
(ii) The union F of all good injections is good.
This is obvious, except for $\mathrm{Ran}(F) \in \mathrm{OR}$. But the alternative $\mathrm{Ran}(F) = \mathrm{OR}$ (cf. Exercise 58) contradicts the Substitution Axiom.
(iii) $\mathrm{Dom}(F) = A$.
If not, let a be the $\prec$-least element of $A - \mathrm{Dom}(F)$ and let $\alpha = \mathrm{Ran}(F)$. Then $F \cup \{(a, \alpha)\}$ is good as well: a contradiction. □

Definition 4.11 The *type* of a well-ordering is the unique ordinal that is isomorphic to it. The type of the well-ordering $(A, \prec)$ is denoted by type$(A, \prec)$. $\square$

Just how far the sequence of ordinals extends remains somewhat of a mystery. Note that every ordinal explicitly named in the sequence displayed on page 23 is countable: every occurrence of '$\dots$' in the sequence represents a countable sequence that has been left out. Nevertheless, uncountable ordinals exist (see Section 4.6), although it probably is impossible for the human mind to obtain a proper image of these things (as is possible for some countable ordinals such as ω, ω^2, ω^ω, $\dots$).

Exercises

61 ♣ Assume that $(A, \prec)$ is a well-ordering and $B \subset A$.
Show that type$(B, \prec) \leqslant$ type$(A, \prec)$.

62 ♣ Show that, for every two non-isomorphic well-orderings, one is isomorphic to a proper initial segment of the other.

63 ♣ Show that there is a limit ordinal $> \omega$.

4.3 Recursion

The first time you see the Recursion Theorem 4.12, it looks terribly abstract. But note that it just is a direct generalization of the natural number case, Theorem 3.16 (p. 16). It is probably best to look at some applications first, and only after that try to understand meaning and proof. These applications often use the simpler type of recursion displayed in Exercise 65. Applications that occur in this text are: Exercise 66 (p. 29), Definition 4.14 (p. 30), Exercise 70 (p. 30), Definition 4.16 (p. 31), Exercise 73 (p. 33), Exercise 76.3 (p. 33), Definition 4.29 (p. 35) and Definition 7.19 (p. 61).

If F is a function and $X \subset \mathrm{Dom}(F)$, then $F|X$ denotes the *restriction* $\{(x, F(x)) \mid x \in X\}$ of F to X.

The recursion equation displayed in Theorem 4.12 expresses that a value $F(\alpha)$ can be calculated, via H, in terms of the initial part $F|\alpha$ of F. Compare the result with Lemma 3.17 (p. 16).

Theorem 4.12 (Recursion on OR**)** *If $H : \mathbf{V} \to \mathbf{V}$ is an operation, then a unique operation $F : \mathrm{OR} \to \mathbf{V}$ exists on* OR *such that for every $\alpha \in \mathrm{OR}$: $F(\alpha) = H(F|\alpha)$.*

Proof. At most one F: transfinite induction.

At least one F: let us call a function f *good* if $\mathrm{Dom}(F) \in \mathrm{OR}$, and f satisfies the recursion equation on its domain: $\forall \alpha \in \mathrm{Dom}(f)(f(\alpha) = H(f|\alpha))$.

As above, it follows that for every two good functions, one must be subset of the other.

It follows that the union F of all good functions is an operation that satisfies the recursion equation on its domain.

It remains to see that $\mathrm{Dom}(F) = \mathrm{OR}$. If not, then $\mathrm{Dom}(F) \in \mathrm{OR}$, and F would be good. Let $\alpha = \mathrm{Dom}(F)$. Then $F \cup \{(\alpha, H(F))\}$ would be good as well; a contradiction. $\square$

There are versions of the recursion theorem with F having parameters. For instance, we might have a recursion equation of the form

$$F(x_1,\dots,x_n,\alpha)=H(x_1,\dots,x_n,\alpha,\{(\beta,F(x_1,\dots,x_n,\beta))\mid\beta<\alpha\}).$$

However, the same proof works.

An abstract version of the recursion theorem holds:

Theorem 4.13 *Suppose that ε is a well-founded relation on the class $\mathbf{U}$ such that for all $a\in\mathbf{U}$, $\{b\in\mathbf{U}\mid b\,\varepsilon\,a\}$ is a set. Then for every operation $H:\mathbf{V}\to\mathbf{V}$ there is a unique operation $F:\mathbf{U}\to\mathbf{V}$ such that for all $a\in\mathbf{U}$:*

$$F(a)=H(F|\{b\in\mathbf{U}\mid b\,\varepsilon\,a\}).$$

Exercises

64 ♣ Prove Theorem 4.13.
Hints. First, assume that ε is transitive. Check that the proof for this special case can be copied, word for word, replacing OR by $\mathbf{U}$, from that of Theorem 4.12.

Next, using this, recursion along a possibly non-transitive ε can be reduced to recursion along its transitive closure $\varepsilon^\star$: Given H, define an auxiliary operation H' by $H'(f)=_{\text{def}}H(f|\{y\mid y\,\varepsilon\,x\})$ if x is such that $\text{Dom}(f)=\{y\mid y\,\varepsilon^\star\,x\}$. (Its values for other arguments are irrelevant.) Now if F satisfies the $\varepsilon^\star$-recursion equation $F(x)=H'(F|\{y\mid y\,\varepsilon^\star\,x\})$, it follows that $F(x)=H'(F|\{y\mid y\,\varepsilon^\star\,x\})=H(F|\{y\mid y\,\varepsilon\,x\})$.

65 ♣ Let $a_0\in\mathbf{V}$ be a set and $G:\mathbf{V}\to\mathbf{V}$ an operation. Show: there exists a unique operation $F:\text{OR}\to\mathbf{V}$ on OR such that

- $F(0)=a_0$,
- $F(\alpha+1)=G(F(\alpha))$,
- for limits γ: $F(\gamma)=\bigcup_{\xi<\gamma}F(\xi)$.

Hint. Apply the Recursion Theorem to a suitable operation H.

66 ♣ Prove Theorem 4.10 using the Recursion Theorem 4.13.
Hint. For the well-ordering $(A,\prec)$, define h on A by recursion along $\prec$ by

$$h(a)=\{h(b)\mid b\in A\ \wedge\ b\prec a\}.$$

Show that h is a 1–1 order preserving function from A to OR, and that $h[A]=\{h(a)\mid a\in A\}$ is the ordinal required. Note that this is just a special case of Exercise 67.

67 ♣ Prove the following generalization of Theorem 4.10:
If ε is well-founded and extensional on the set A, then there is a unique transitive set B such that $(B,\in)\cong(A,\varepsilon)$.
(This is called *Mostowski's Collapsing Lemma*, cf. Lemma 7.52 p. 73. Erasing the well-foundedness condition results in a statement —an example of an *Anti-Foundation Axiom*— that contradicts the Foundation Axiom.)
Generalize to the following theorem: If ε, next to satisfying the conditions from Theorem 4.13, is *extensional* on the class $\mathbf{U}$ (elements in $\mathbf{U}$ with the same ε-predecessors are the same), then there is a unique transitive class T such that $(\mathbf{U},\varepsilon)\cong(T,\in)$.

68 ♣ Prove the following special case of the Collection Principle: if $\forall x\in a\exists\alpha\in\text{OR}\Phi(x,\alpha)$, then $\beta\in\text{OR}$ exists such that $\forall x\in a\,\exists\alpha\in\beta\,\Phi(x,\alpha)$.

4.4 Fixed Point Hierarchies

Compare Theorem 3.27 (p. 20).

Definition 4.14 Let H be a monotone operator over the class $\mathbf{U}$ satisfying the property that $X \in \mathbf{V} \Rightarrow H(X) \in \mathbf{V}$ ("if X is a set, then so is $H(X)$" — this is needed for the following hierarchy to exist). The *least fixed point hierarchy* asociated with H is the sequence $\{H{\uparrow}\alpha\}_{\alpha\in\mathrm{OR}}$ of *stages* $H{\uparrow}\alpha$, recursively defined by

- $H{\uparrow}0 = \emptyset$
- $H{\uparrow}(\alpha+1) = H(H{\uparrow}\alpha)$
- $H{\uparrow}\gamma = \bigcup_{\xi<\gamma} H{\uparrow}\xi$ (for limits γ).

If for some ordinal α, $H{\uparrow}\alpha$ coincides with the least fixed point $H{\uparrow}$ of H (equivalently, for all $\beta > \alpha$, $H{\uparrow}\beta = H{\uparrow}\alpha$), then the least such ordinal α is called the *closure ordinal* of the hierarchy. □

Let us call H *$\star$-finite* if, whenever $a \in H(X)$, there is a sub*set* $x \subset X$ such that $a \in H(x)$.

Theorem 4.15 *Suppose that H is $\star$-finite, monotone over $\mathbf{U}$, and maps sets to sets. Then:*

1. *the least fixed point hierarchy associated with H is* cumulative*: $\alpha < \beta \Rightarrow H{\uparrow}\alpha \subset H{\uparrow}\beta$,*
2. *$\bigcup_\alpha H{\uparrow}\alpha$ is the least fixed point $H{\uparrow}$ of H,*
3. *if $\mathbf{U}$ is a set, then a closure ordinal exists.*

Proof. 1. Induction w.r.t. β.

2. Closure, $H(\bigcup_\alpha H{\uparrow}\alpha) \subset \bigcup_\alpha H{\uparrow}\alpha$:
Assume that $a \in H(\bigcup_\alpha H{\uparrow}\alpha)$. Since H is $\star$-finite, we have that $a \in H(X)$ for some subset $X \subset \bigcup_\alpha H{\uparrow}\alpha$. So, $\forall x \in X\, \exists \alpha \in \mathrm{OR}\, x \in H{\uparrow}\alpha$. By Exercise 68, for some $\beta \in \mathrm{OR}$, $\forall x{\in}X\, \exists\alpha{\in}\beta\, x \in H{\uparrow}\alpha$. Thus, $X \subset H{\uparrow}\beta$, and $a \in H(X) \subset H(H{\uparrow}\beta) = H{\uparrow}(\beta+1) \subset \bigcup_\alpha H{\uparrow}\alpha$.
Induction, $H(X) \subset X \Rightarrow \bigcup_\alpha H{\uparrow}\alpha \subset X$:
Assume that $H(X) \subset X$. By induction on α it follows that $H{\uparrow}\alpha \subset X$.

3. If $\mathbf{U}$ is a set and a closure ordinal does not exist, then the least fixed point hierarchy (the map $\alpha \mapsto H{\uparrow}\alpha$) constitutes an injection of OR into $\wp(\mathbf{U})$. This contradicts Exercise 14 p. 9. □

By the argument for 4.15.3, the closure ordinal has power at most $|\wp(\mathbf{U})|$. From the *Axiom of Choice* it even follows that the closure ordinal has power at most $|\mathbf{U}|$: cf. Exercise 106 (p. 40).

Exercises

69 ♣ Show that, for $\alpha \in \mathrm{OR}$, $\mathbf{T}{\uparrow}\alpha = \alpha$. (N.B.: $\mathbf{T}$ is the operator that generates OR.)

70 ♣ Show that the *single* recursion equation $H{\uparrow}\alpha = \bigcup_{\xi<\alpha} H(H{\uparrow}\xi)$ defines the same operation as the one defined in Definition 4.14 by *three* equations. (And, of course, $H{\downarrow}\alpha = \bigcap_{\xi<\alpha} H(H{\downarrow}\xi)$ is a single equation defining the greatest fixed point hierarchy — cf. Exercise 72.)

71 ♣ Do not assume that the $\star$-finite monotone H maps sets to sets. Show that a hierarchy of *classes* $\{H{\uparrow}\alpha\}_\alpha$ is definable that satisfies the properties of Definition 4.14. Prove a version of Theorem 4.15 for this case.

72 ♣ Let H be a monotone operator over a *set* $\mathbf{U}$. The *greatest fixed point hierarchy* is the sequence $\{H{\downarrow}\alpha\}_\alpha$ recursively defined by

- $H{\downarrow}0 = \mathbf{U}$,
- $H{\downarrow}(\alpha + 1) = H(H{\downarrow}\alpha)$,
- $H{\downarrow}\gamma = \bigcap_{\xi<\gamma} H{\downarrow}\xi$ (for limits γ).

Show that:

1. the hierarchy is *descending*, i.e., that $\alpha < \beta \Rightarrow H{\downarrow}\beta \subset H{\downarrow}\alpha$.
2. some stage $H{\downarrow}\alpha_0$ is a fixed point of H.
3. $H{\downarrow}\alpha_0 = \bigcap_\alpha H{\downarrow}\alpha$ is the greatest fixed point of H.

Try to generalize for the case where $\mathbf{U}$ may be a proper class.

4.5 Cumulative Hierarchy

The *cumulative hierarchy* —already pointed at in Chapter 1— is the least fixed point hierarchy of the powerclass operator $\wp$ over $\mathbf{V}$, where

$$\wp(X) = \{x \in \mathbf{V} \mid x \subset X\}.$$

Note that this operator maps sets to sets (Powerset Axiom) and is $\star$-finite (if $a \in \wp(X)$, then $a \subset X$ and of course $a \in \wp(a)$). Thus, Theorem 4.15 applies.

Stages $\wp{\uparrow}\alpha$ are called *partial universes* and usually denoted V_α (sometimes R_α). There is no closure ordinal here. (Why?)

Definition 4.16

- $\mathrm{V}_0 = \emptyset$,
- $\mathrm{V}_{\alpha+1} = \wp(\mathrm{V}_\alpha)$,
- $\mathrm{V}_\gamma = \bigcup_{\xi<\gamma} \mathrm{V}_\xi$ (for limits γ). □

The sets V_α are called *partial universes* and the sequence of partial universes is called the *cumulative hierarchy*, cf. Chapter 1.

Lemma 4.17

1. *Every* V_α *is transitive,*
2. $x \subset y \in \mathrm{V}_\alpha \;\Rightarrow\; x \in \mathrm{V}_\alpha$*,*
3. $\alpha < \beta \;\Rightarrow\; \mathrm{V}_\alpha \in \mathrm{V}_\beta$*;* $\alpha \leqslant \beta \;\Rightarrow\; \mathrm{V}_\alpha \subset \mathrm{V}_\beta$*,*

4. $\alpha \subset \mathrm{V}_\alpha$; $\alpha \notin \mathrm{V}_\alpha$; $\alpha = \mathrm{OR} \cap \mathrm{V}_\alpha$,

5. $\mathrm{OR} \cap (\mathrm{V}_{\alpha+1} - \mathrm{V}_\alpha) = \{\alpha\}$.

The following results "explain" the Foundation Axiom.

Theorem 4.18 $\mathbf{G}$ *is the least fixed point of* $\wp$ *(and* $\mathbf{V}$ *is the greatest one).*

Proof. 1. $\wp(\mathbf{G}) \subset \mathbf{G}$. See Exercise 16 (p. 10).
2. $\wp(X) \subset X \;\Rightarrow\; \mathbf{G} \subset X$:
Aiming at a contradiction, assume that $\wp(X) \subset X$, $a \in \mathbf{G}$, $a \notin X$.
Claim: $\mathrm{TC}(a) \subset X$.
(And it follows that $a \subset X$ and $a \in X$: contradiction.)
Proof: Assume that $\mathrm{TC}(a) - X \neq \emptyset$. Note that (since $\mathbf{G}$ is transitive: Exercise 16) $a \subset \mathbf{G}$ and hence $\mathrm{TC}(a) \subset \mathbf{G}$ (again Exercise 16). Thus, if some b is $\in \mathrm{TC}(a) - X$, it is in $\mathbf{G}$ as well; and therefore some $b \in \mathrm{TC}(a) - X$ has $b \cap (\mathrm{TC}(a) - X) = \emptyset$. But then $b \subset X$ and $b \in X$. □

Corollary 4.19 *The following are equivalent:*

1. *The Foundation Axiom,*

2. $\bigcup_\alpha \mathrm{V}_\alpha = \mathbf{V}$,

3. *every non-empty class is disjoint with some of its elements,*

4. *($\in$-Induction) for every class* K*, if* $\forall a\,[\forall b(b \in a \Rightarrow b \in K) \Rightarrow a \in K]$*, then* $\mathbf{V} \subset K$.

Proof. $1 \Leftrightarrow 2$. Immediate, since Foundation says that $\mathbf{V} = \mathbf{G}$, Theorem 4.18 says that $\mathbf{G} = \wp\!\Uparrow$, and Theorem 4.15.2 implies that $\wp\!\Uparrow = \bigcup_\alpha \mathrm{V}_\alpha$.

$1 \Rightarrow 4$. Note that $\in$-induction is nothing but the implication $\wp(K) \subset K \;\Rightarrow\; \mathbf{V} \subset K$. By Theorem 4.18, this holds when $\mathbf{V}$ is replaced by $\mathbf{G}$. However, Foundation says that $\mathbf{V} = \mathbf{G}$.

$3 \Leftrightarrow 4$. These principles are logically equivalent (take complements of the classes involved).

$3 \Rightarrow 1$. Trivial, since every set is a class. □

Definition 4.20 For $a \in \bigcup_\alpha \mathrm{V}_\alpha$, define the *rank* $\rho(a)$ of a by: $\rho(a) := \bigcap\{\alpha \mid a \subset \mathrm{V}_\alpha\}$. □

If Foundation is assumed, you can define an operation on non-empty classes that uniformly selects a non-empty subset.

Definition 4.21 $\mathrm{Bottom}(X) := \{x \in X \mid \forall y \in X(\rho(x) \leqslant \rho(y))\}$. □

Lemma 4.22 1. $\mathrm{Bottom}(X) \subset X$,

2. $\mathrm{Bottom}(X) \in \mathbf{V}$,

3. *if* $X \neq \emptyset$*, then* $\mathrm{Bottom}(X) \neq \emptyset$.

Exercises

73 ♣ Show that $\mathrm{V}_\alpha = \bigcup_{\xi<\alpha} \wp(\mathrm{V}_\xi)$. (This is a *single* recursion equation defining the sequence $\{\mathrm{V}_\xi\}_\xi$. Cf. Exercise 70. You better prove Lemma 4.17.1 first.)

74 ♣ Show that the class $\bigcup_\alpha \mathrm{V}_\alpha$ is the least fixed point of the powerclass operation $X \mapsto \{a \in \mathbf{V} \mid a \subset X\}$.

75 ♣ Prove Lemma 4.17.
Hint. Use the equation from Exercise 73. Every single item of the lemma has a one-line proof.

76 ♣ Show:

1. $\rho(\alpha) = \rho(\mathrm{V}_\alpha) = \alpha$,
2. $\mathrm{V}_\alpha = \{a \mid \rho(a) < \alpha\}$; $a \in b \;\Rightarrow\; \rho(a) < \rho(b)$,
3. $\rho(a) = \bigcup\{\rho(b)+1 \mid b \in a\} = \{\rho(b) \mid b \in \mathrm{TC}(a)\}$
   (these can be looked at as recursions —along $\in$, resp., $\in^\star$— that define ρ),
4. express $\rho(a \cup b)$, $\rho(\bigcup a)$, $\rho(\wp(a))$, $\rho(\{a\})$, $\rho((a,b))$ and $\rho(\mathrm{TC}(a))$ in terms of $\rho(a)$ and $\rho(b)$.

77 ♣ Suppose that the operation $W : \mathrm{WF} \to \mathbf{V}$ satisfies the ($\in$-recursion) equation $W(a) = \bigcup_{b\in a} \wp W(b)$. Show that $W(a) = \mathrm{V}_{\rho(a)}$.

78 ♣ Assuming the Foundation Axiom, prove the Collection Principle:
$\forall x \in a\, \exists y\, \Phi(x,y) \;\Rightarrow\; \exists b\, \forall x \in a\, \exists y \in b\, \Phi(x,y)$ (b not free in Φ).
Hint. For $x \in a$, let $h(x)$ be the least ordinal α such that $\exists y \in \mathrm{V}_\alpha \Phi(x,y)$. Let $b := \mathrm{V}_\beta$, where $\beta = \bigcup\{h(x) \mid x \in a\}$ is the *sup* of all the $h(x)$ ($x \in a$).
Or, apply Exercise 68.

79 ♣ Assuming the Foundation Axiom, prove the Strong Collection Principle:
$\forall x \in a\, \exists y\, \Phi(x,y) \;\Rightarrow\; \exists b\, [\forall x \in a\, \exists y \in b\, \Phi(x,y) \wedge \forall y \in b\, \exists x \in a\, \Phi(x,y)]$ (b not free in Φ).

80 ♣ Suppose that R is a binary property on a non-empty class X such that $\forall x \in X \exists y \in X(xRy)$. Using Foundation, show that a non-empty set $A \subset X$ exists such that $\forall x \in A \exists y \in A(xRy)$. (Thus, the "class-form" of DC is implied by DC.)

The next two exercises demonstrate that the relations $\rho(x) \leqslant \rho(y)$ and $\rho(x) < \rho(y)$ can be defined without reference to partial universes or, even, to ordinals.

81 ♣ Define the monotone operator Γ (that maps binary properties to binary properties) by $\Gamma(R) := \{(x,y) \mid \forall x' \in x \exists y' \in y\, x'Ry'\}$.
Show that (n.b.: $\mathrm{WF} = \bigcup_\alpha \mathrm{V}_\alpha$) $x\Gamma{\Uparrow}y$ iff both $x \in \mathrm{WF}$ and $y \in \mathrm{WF} \Rightarrow \rho(x) \leqslant \rho(y)$.

82 ♣ Find a monotone operator Γ' for which $x\Gamma'{\Uparrow}y$ iff both $x \in \mathrm{WF}$ and $y \in \mathrm{WF} \Rightarrow \rho(x) < \rho(y)$.

The next exercise uses the finite partial universes as a surrogate for the natural numbers; the next two exercises contain simple definitions of the notion of a partial universe.

83 ♣

1. Suppose that the set r is such that $\forall x[x \in r \;\Rightarrow\; \exists y \subset r(x = \bigcup_{z \in y} \wp(z))]$. Show that every element of r is a partial universe.

2. Show that a set is a partial universe iff it is an element of a set r with the property that $\forall x[x \in r \;\Rightarrow\; \exists y \subset r(x = \bigcup_{z \in y} \wp(z))]$.

3. Show that a set is a *finite* partial universe iff it is an element of a set r with the property that $\forall x[x \in r \;\Rightarrow\; \exists y \in r(x = \wp(y))]$.

84 ♣ *Scott's 1967-axiomatization* of set theory in [Scott 74] uses a new primitive notion of *partial universe* (*p.u.*) and consists of the Axioms of Extensionality and Separation, the statement that every p.u. is a set, the *Accumulation Axiom* $x \in V \leftrightarrow \exists V' \in V(x \subset V')$ (where V and V' are variables for p.u.'s — Scott's formulation is slightly more complicated, allowing for non-sets) and the *Restriction Axiom* $\forall x \exists V(x \subset V)$. Using these axioms, derive the Foundation Axiom and the Sumset Axiom. Prove that a set that is element of a set has a powerset.

(Note that Restriction in a sense simulates Lemma 4.19; Accumulation is nothing but the recursion from Exercise 73 where the role of the ordinals is taken over by the p.u.'s themselves.)

The point of the following exercises is to describe (some of) the relationship between the structures $(\mathrm{V}_\omega, \in)$ and $(\mathbb{N}, <)$ (i.e., $(\omega, <)$). Although $\mathbb{N}$ forms only a tiny part of V_ω, the latter one happens to be represented, neatly coded, in $\mathbb{N}$.

First, recursively define $h : \mathrm{V}_\omega \to \mathbb{N}$ by

$$h(x) = \sum_{y \in x} 2^{h(y)}.$$

85 ♣ Show that h is a bijection.

The function h transforms $\in$ on V_ω into the relation ε on $\mathbb{N}$ for which

$n \varepsilon m$ iff the binary notation for n has a 1 in the $(m + 1)$-st position.

E.g., since 5 $(= 2^2 + 2^0)$ has binary notation 101, it follows that 0 and 2 are the only ε-predecessors of 5.

Conversely, the ordering $<$ of $\mathbb{N}$ is transformed into a relation $\prec$ on V_ω.

86 ♣ Show: there is exactly one relation $\prec$ on WF such that for all $x, y \in$ WF:

$$x \prec y \;\Leftrightarrow\; \exists y' \in y - x \, \forall x' \in x - y \, (x' \prec y').$$

(This modifies the fixed point conditions from Exercises 81 and 82.)

87 ♣ Show that for $x, y \in \mathrm{V}_\omega$: $x \prec y$ iff $h(x) < h(y)$.
In particular, $\prec$ is a well-ordering of V_ω. Show that the V_n are initials in this ordering. What does it do outside V_ω?

4.6 Initial Numbers

See the beginning of Section 4.1 (p. 23): note that all ordinals of the initial segement of OR that is sketched there are *countable*. This section starts with proving that *uncountable ordinals* exist.

Definition 4.23

1. $A \leqslant_1 B :\equiv$ there exists an injection : $A \to B$,

2. $A =_1 B :\equiv$ there exists a bijection : $A \to B$,

3. $A <_1 B :\equiv A \leqslant_1 B \wedge \neg A =_1 B$. □

Definition 4.24 A is *countable* iff $A \leqslant_1 \omega$; A is *countably infinite* iff A is countable and infinite. (Cf. Definition 3.15 p. 15.) □

Definition 4.25 (Hartogs' operation.) $\Gamma(A) := \{\alpha \mid \alpha \leqslant_1 A\}$. □

Lemma 4.26 *$\Gamma(A)$ is the least ordinal α such that $\neg\alpha \leqslant_1 A$.*
In particular, if $\beta \in$ OR, then $\Gamma(\beta)$ is the least ordinal α such that $\beta <_1 \alpha$.

Proof. $\Gamma(A)$ is easily seen to be a *transitive* class of ordinals. To see that it is an ordinal, it therefore suffices to show that it is a *set*. But this follows (using the Axioms of Powerset, Separation, and Substitution) from the fact, that $\Gamma(A) = \{\text{type}(X, R) \mid X \subset A \wedge R$ well-orders $X\}$. Finally, if $\Gamma(A) \leqslant_1 A$, then $\Gamma(A) \in \Gamma(A)$; hence, $\neg\Gamma(A) \leqslant_1 A$. And if $\alpha < \Gamma(A)$, then $\alpha \leqslant_1 A$. □

Without Axiom of Choice, it is unprovable that $A <_1 \Gamma(A)$ and $\Gamma(A) \leqslant_1 \wp(A)$. In this connection, see Exercise 89.

Definition 4.27 An *initial number* is an ordinal $\alpha \geqslant \omega$ such that $\forall\xi < \alpha(\xi <_1 \alpha)$.

The equivalence $=_1$ partitions OR in *number classes*. The classes of natural numbers are singletons. Then follows the (uncountable) class of ω: the countably infinite ordinals which are $\geqslant \omega$ and $< \Gamma(\omega)$. $\Gamma(\omega)$ is the first uncountable ordinal in a long series of ordinals of the same power, etc.

Lemma 4.28 *1. ω is the least initial.*

2. If $\omega \leqslant \alpha$, then $\Gamma(\alpha)$ is the least initial $> \alpha$.

3. If $\alpha \leqslant \beta < \Gamma(\alpha)$, then $\beta =_1 \alpha$.

4. Every infinite ordinal is $=_1$ to an initial.

Definition 4.29 Using recursion on OR, define the series ω_α as follows:

1. $\omega_0 = \omega$,

2. $\omega_{\alpha+1} = \Gamma(\omega_\alpha)$,

3. for limits γ: $\omega_\gamma = \bigcup_{\xi<\gamma} \omega_\xi$. □

Lemma 4.30 *The operation ω_α enumerates the class of initials. I.e.,*

1. *every ω_α is an initial,*
2. *every initial has the form ω_α,*
3. $\alpha < \beta \;\Rightarrow\; \omega_\alpha < \omega_\beta$.

Exercises

88 ♣ Show: A is finite (Definition 3.15 p. 15) iff $A <_1 \omega$.

89 ♣ $\Gamma(A) \leqslant_1 \wp(\wp(A \times A))$.

90 ♣ Prove Lemma 4.28. (You may need the Cantor-Bernstein Theorem 6.6 p. 42.)

91 ♣ Prove Lemma 4.30.

92 ♣ Show that every initial is a limit.

Although at first sight, $\alpha << \omega_\alpha$, from time to time, initials catch up with their index. This occurs for the first time at an initial that is quite large:

93 ♣ Let $\alpha \in$ OR be arbitrary. Recursively define $\alpha_0 = \alpha$ and $\alpha_{n+1} = \omega_{\alpha_n}$. Put $\beta := \bigcup_n \alpha_n$. Show: β is the least ordinal $\gamma \geqslant \alpha$ for which $\omega_\gamma = \gamma$.

4.7 Arithmetic

The operations of addition, multiplication and exponentiation can be extended to OR by the following definition.

Definition 4.31 (In the following, γ is an arbitrary limit.)

1. $\alpha + 0 = \alpha$;
   $\alpha + (\beta + 1) = (\alpha + \beta) + 1$;
   $\alpha + \gamma = \bigcup_{\xi<\gamma}(\alpha + \xi)$,
2. $\alpha \cdot 0 = 0$;
   $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \beta$;
   $\alpha \cdot \gamma = \bigcup_{\xi<\gamma}(\alpha \cdot \xi)$,
3. $\alpha^0 = 1$;
   $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$;
   $\alpha^\gamma = \bigcup_{\xi<\gamma} \alpha^\xi$. □

Warning. The notation A^B may also denote $\{f \mid f : B \to A\}$ (see Definition 6.7 p. 42 and the warning there). Thus, if A and B are ordinals, A^B is ambiguous. (Some writers prefer the notation ${}^BA$ for the set of functions.)

See elsewhere, for instance [Levy 77], for the basic properties of these operations, and for the following.

Theorem 4.32 (Cantor Normal Form) *Fix $\theta > 1$. Every ordinal α can be written in exactly one way as $\alpha = \theta^{\beta_1} \cdot \gamma_1 + \cdots + \theta^{\beta_k} \cdot \gamma_k$ with $\beta_1 > \cdots > \beta_k$ and $\gamma_1, \ldots, \gamma_k < \theta$.*

The Cantor Normal Form Theorem for $\theta = 10$ justifies our use of decimal notations for natural numbers. For $\theta = \omega$ it offers "notations" (in terms of natural numbers) for all ordinals below $\varepsilon_0 := \bigcap\{\alpha \mid \omega^\alpha = \alpha\} = \bigcup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$. This is used in Gentzen's consistency proof for Peano arithmetic.

Definition 4.33 α is called *critical* for the operation $F : \mathrm{OR}^2 \to \mathrm{OR}$ if $\beta, \gamma < \alpha \Rightarrow F(\beta, \gamma) < \alpha$. □

Exercises

94 ♣ Show: $\alpha \cdot \omega$ is the least ordinal $> \alpha$ that is critical for $+$.

95 ♣ For $\alpha \geqslant \omega$, the following are equivalent:
1. α is critical for $+$; 2. $\beta < \alpha \Rightarrow \beta + \alpha = \alpha$; 3. $\exists \xi\ (\alpha = \omega^\xi)$.

96 ♣ Assume $\alpha \geqslant \omega$.
Show: α^ω is the least ordinal $> \alpha$ that is critical for multiplication.

97 ♣ Show that, for $\alpha \geqslant \omega$, the following are equivalent:
1. α is critical for multiplication; 2. $\beta < \alpha \Rightarrow \beta \cdot \alpha = \alpha$; 3. $\exists \xi\ (\alpha = \omega^{\omega^\xi})$.

98 ♣ Show: every initial is critical for addition, multiplication and exponentiation.

4.8 Well-ordering of $\mathrm{OR} \times \mathrm{OR}$

Definition 4.34 Define the relation $<$ on $\mathrm{OR} \times \mathrm{OR}$ by:
$(\alpha, \beta) < (\alpha', \beta') :\equiv$
$\max(\alpha, \beta) < \max(\alpha', \beta')$, or: $\max(\alpha, \beta) = \max(\alpha', \beta') \wedge [\alpha < \alpha' \vee (\alpha = \alpha' \wedge \beta < \beta')]$.
□

99 ♣ Exercise Show:

1. $<$ well-orders $\mathrm{OR} \times \mathrm{OR}$,
2. every product $\gamma \times \gamma$ is an initial segment
   (if $(\alpha, \beta) < (\alpha', \beta') \in \gamma \times \gamma$, then $(\alpha, \beta) \in \gamma \times \gamma$),
3. the product $\omega \times \omega$ is well-ordered in type ω,
4. every product $\omega_\alpha \times \omega_\alpha$ $(\alpha > 0)$ is well-ordered in type ω_α.

Hint. 4 says that, if $\Gamma : \mathrm{OR} \times \mathrm{OR} \to \mathrm{OR}$ is the unique order-preserving map, then $\Gamma(\omega_\alpha, \omega_\alpha) = \omega_\alpha$. Use induction w.r.t. α. If equality doesn't hold, then we must have $\Gamma(\omega_\alpha, \omega_\alpha) > \omega_\alpha$. Then $(\beta, \gamma) \in \omega_\alpha \times \omega_\alpha$ exists such that $\Gamma(\beta, \gamma) = \omega_\alpha$, etc.

Part 4 of this exercise has the important

Corollary 4.35 $\omega_\alpha \times \omega_\alpha =_1 \omega_\alpha$. □

Chapter 5

Axiom of Choice

Definition 5.1 Let A be a set (of sets). The function f is a *choice function* for A if $\mathrm{Dom}(f) = A - \{\emptyset\}$ and $\forall x \in \mathrm{Dom}(f)\ (f(x) \in x)$. □

The **Axiom of Choice** is the following statement:

Every set has a choice function.

AC means Axiom of Choice. ZFC is ZF together with AC.

For an introduction to AC, read the appropriate parts of [Doets 97]. For the history of the axiom, cf. [Moore 82]. For an overwhelming number of equivalents, [Rubin & Rubin 63]. For metamathematics, [Jech 73]. In 1936, Gödel showed that AC is consistent with the ZF-axioms (cf. Chapter 7); in 1963, Cohen showed it to be independent. The following is an ultra-short treatment.

Of course, sometimes you do not need AC to see that a choice function exists; for an example, see Exercise 100.

Here follow the most important equivalents of AC.

Well-ordering Theorem (Zermelo 1904/08): *Every set has a well-ordering.*

Enumeration Theorem: *Every set is $=_1$ to an ordinal.*

Trichotomy Theorem (Hartogs 1915): *For every two sets A and B we have that $A =_1 B$ or $A <_1 B$ or $B <_1 A$.*

Zorn's Lemma (Zorn 1935): *If $\preceq$ partially orders the set A in such a way that every* chain *(that is: a by $\preceq$ linearly ordered subset of A) has an upper bound, then A has at least one $\preceq$-maximal element.*

Here follow (sketches of) proofs of equivalence.

Enumeration implies Well-ordering
If $f : A \to \alpha \in \mathrm{OR}$ is bijective, then $\{(x, y) \in A^2 \mid f(x) < f(y)\}$ well-orders A. □

Well-ordering implies Enumeration
If $\prec$ well-orders the set A, then $A =_1 \mathrm{type}(A, \prec) \in \mathrm{OR}$. □

Enumeration implies Trichotomy
If $A =_1 \alpha$ and $B =_1 \beta$ and, say, $\alpha \leqslant \beta$, then $A \leqslant_1 B$. (Hence, $A <_1 B$ or $A =_1 B$.) □

Trichotomy implies Well-ordering
Let A be any set. Since $\neg\Gamma(A) \leqslant_1 A$, by Trichotomy we have $A \leqslant_1 \Gamma(A)$, etc. □

Well-ordering implies AC
Let $\prec$ be a well-ordering of $\bigcup A$. A choice function for A is the function that, with every non-empty $a \in A$, associates the $\prec$-least element of a. $\square$

Zorn implies Trichotomy
Let A and B be arbitrary sets. We want an injection : $A \to B$, or one : $B \to A$. The crux is to find a partial ordering in which such injections are maximal objects. Define $F =_{\text{def}} \{f \mid f$ is an injection such that $\text{Dom}(f) \subset A$ and $\text{Ran}(f) \subset B\}$. Consider the partial ordering $\subset$ of F. $\square$

AC implies Zorn
Assume that $\preceq$ is a partial ordering of the set A in which linearly ordered subsets have upper bounds. Assume that A has no maximal element. Let f be a choice function for $\wp(A)$. Recursively define $H : \text{OR} \to A$ such that $H(\alpha) = f(\{a \in A \mid \forall \xi < \alpha\,(H(\xi) \prec a)\})$ in case that $\{a \in A \mid \forall \xi < \alpha\,(H(\xi) \prec a)\} \neq \emptyset$ and $H(\alpha) = A$ otherwise. Note that, in fact, $\forall \alpha (H(\alpha) \in A)$ and $\forall \xi \forall \alpha (\xi < \alpha \Rightarrow H(\xi) \prec H(\alpha))$. (Induction. Assume as IH: if $\delta < \alpha$, then $\forall \xi \leqslant \delta (H(\xi) \in A)$ and $\xi < \delta \Rightarrow H(\xi) \prec H(\delta)$. Then $\{H(\xi) \mid \xi < \alpha\}$ is linearly ordered, and has an upper bound. Since this upper bound cannot be maximal, $\{a \in A \mid \forall \xi < \alpha\,(H(\xi) \prec a)\} \neq \emptyset$; and hence $H(\alpha) \in A$ and $\xi < \alpha \Rightarrow H(\xi) \prec H(\alpha)$.) Thus, H injects OR into A, contradicting the Substitution Axiom. $\square$

Exercises

100 ♣ Show, without using AC, that every *finite* set has a choice function.

101 ♣ The *Axiom of Dependent Choice* (DC) says: if the set A is non-empty and the relation $R \subset A^2$ is such that $\forall a \in A \exists b \in A (aRb)$, then a function $f : \omega \to A$ exists such that for all $n \in \omega$, $f(n) R f(n+1)$.

1. Assume AC. Prove DC.
2. Show the version of DC where A can be a proper class and $R \subset A^2$ is also provable from AC. (Use Foundation.)
3. Show that a relation $\prec$ is well-founded (every non-empty set has a $\prec$-minimal element) iff there is no function f on ω such that for all $n \in \omega$, $f(n+1) \prec f(n)$.

102 ♣ (AC) Show: $\bigcup_{i \in I} A_i \leqslant_1 \bigcup_{i \in I} A_i \times \{i\}$.

103 ♣ (AC) Show: if A is infinite, then $\omega \leqslant_1 A$.
Show *without* AC that: if A is infinite, then $\omega \leqslant_1 \wp(\wp(A))$.

104 ♣ (H. Rubin) Assume the Foundation Axiom. Show: AC is equivalent with the statement that powersets of ordinals have well-orderings.
(So, under the Foundation Axiom, GCH $\Rightarrow$ AC (cf. Section 6.5 p. 50). Note: when GCH is suitably formulated, this implication is provable even without Foundation.)
Hint. Let α be arbitrary. Put $\lambda =_{\text{def}} \Gamma(\text{V}_\alpha)$. Using a well-ordering of $\wp(\lambda)$, recursively define well-orderings for all V_ξ, $\xi \leqslant \alpha$.

105 ♣ Show that the following are equivalent for every two sets A and B:

1. $A <_1 B$, i.e.: there is no bijection : $A \to B$ and $A \leqslant_1 B$,
2. there is no surjection : $A \to B$ and $A \leqslant_1 B$,
3. there is no surjection : $A \to B$ and $B \neq \emptyset$.

For which of the six implications do you need AC?

106 ♣ Show that the closure ordinal (cf. Definition 4.14 p. 30) of a monotone operator over a set A has power at most $|A|$.

107 ♣ Give direct proofs for the following implications.

1. AC implies the Well-ordering Theorem. (N.B.: it is worthwile looking up Zermelo's beautiful original proofs which do not use ordinals and recursion.)
2. AC implies the Enumeration Theorem.
3. Zorn implies AC.
4. Zorn implies the Well-ordering Theorem.

Hints. 2. Compare the proofs for Theorems 4.10 (p. 27) and 4.12 (p. 28). In order to construct a well-ordering for a set A, fix a choice function $h : \wp(A) \to A$ for $\wp(A)$. A function f is *good* if $\mathrm{Dom}(f) \in \mathrm{OR}$, $\mathrm{Ran}(f) \subset A$, and for all $\alpha \in \mathrm{Dom}(f)$, $\mathrm{Ran}(f|\alpha)$ is a proper subset of A, and $f(\alpha) = h(A - \mathrm{Ran}(f|\alpha))$. Show that the union F of all good functions injects $\mathrm{Dom}(F)$ into A. Note that $\mathrm{Dom}(F) \neq \mathrm{OR}$ because of the Substituton Axiom. Thus, we have $\mathrm{Dom}(F) \in \mathrm{OR}$ and F is good. Also, $\mathrm{Ran}(F) = A$, for if not —say, $a \in A - \mathrm{Ran}(F)$—, we could properly extend F to $F \cup \{(\mathrm{Dom}(F), a)\}$, which would be another good function: a contradiction.

1. We can eliminate the use of ordinals from this argument, directly constructing the well-ordering that corresponds to the above enumeration, as follows. Say that $(B, \prec)$ is *good* if $B \subset A$, $\prec$ well-orders B, and for all $b \in B$: $b = h(A - \{a \in A \mid a \prec b\})$. Show that the union of all good well-orderings is a well-ordering for (all of) A. (This is one of Zermelo's proofs. Note that the Substitution Axiom is not needed here.)

Warning, also w.r.t. part 4: It is *not* true that a union of a class G of well-orderings that is linearly ordered under unclusion (i.e.: for which $\forall R, S \in G(R \subset S \vee S \subset R)$) is always a well-ordering. (Can you think of an example?) The point is that, for every two good well-orderings, one must be an *initial segment* of the other one.

108 ♣ The *Teichmüller-Tukey Lemma* is the following statement.

> *Suppose that $\emptyset \neq A \subset \wp(X)$, and for all $Y \subset X$, Y is in A iff every finite subset of Y is in A. Then A has a ($\subset$-) maximal element.*

Show that this is equivalent with Zorn's Lemma.
Hint. To prove that this implies Zorn, assume that $(X, \preceq)$ is a partial ordering, and let A be the set of (by $\preceq$) linearly ordered subsets of X.

Chapter 6

Cardinals

6.1 Definition

The *cardinal* (*cardinal number*) of a set A is an object $|A|$ such that the following equivalence is satisfied:

$$|A| = |B| \Leftrightarrow A =_1 B.$$

($A =_1 B$ means that a bijection between A and B exists —See Definition 4.23.2 p. 35.)
A *cardinal* (*number*) is a cardinal (number) of a set.

The *Frege-Russell definition* of cardinals is $|A| =_{\text{def}} \{B \mid B =_1 A\}$. This satisfies the equivalence. However, $|A|$ is a set only if $A = \emptyset$. Assuming AC, the following is a definition of $|A|$ as a set such that the above equivalence is provable:

Definition 6.1 $|A| =_{\text{def}} \bigcap\{\alpha \in \text{OR} \mid \alpha =_1 A\}$. □

This defines $|A|$ to be a *canonical element* of the Frege-Russell cardinal of A.

If the Regularity Axiom is available, the following definition can be used also (Scott; see Lemma 4.22 p.32):

Definition 6.2 $|A| =_{\text{def}} \text{Bottom}(\{B \mid B =_1 A\})$
$(= \{B \mid B =_1 A \wedge \forall C\,(C =_1 A \Rightarrow \rho(B) \leqslant \rho(C))\})$. □

This makes $|A|$ a *canonical selection* of the Frege-Russell cardinal of A. If neither AC nor Foundation are available, a definition of cardinality satisfying the required equivalence is not possible (Levy). In that situation one solution remains: consider the operation $|\,|$ as a *primitive notion* and the above equivalence as an *axiom*.

For the following, it is irrelevant how cardinals have been introduced, as long as this equivalence is satisfied.

6.2 Elementary Properties and Arithmetic

Definition 6.3 $\aleph_\alpha =_{\text{def}} |\omega_\alpha|$; an *aleph* is a cardinal of the form $\aleph_\alpha$. □

AC is equivalent with the statement that every cardinal is an aleph.

Without loss of generality you may assume that 6.1 is satisfied whenever A has a well-ordering. In that case:

Corollary 6.4 $\aleph_\alpha = \omega_\alpha$; $n \in \omega \;\Rightarrow\; |n| = n$. □

Definition 6.5 Define the relations $<$ and $\leqslant$ between cardinals such that:

1. $|A| \leqslant |B| \;\Leftrightarrow\; A \leqslant_1 B$ (i.e.: there is an injection : $A \to B$),
2. $|A| < |B| \;\Leftrightarrow\; A <_1 B$ (i.e.: there is an injection : $A \to B$ but no bijection).

I.e., (e.g., in the case of 1): for p, q cardinals, $p \leqslant q$ should be defined as: there are sets A, B exist s.t. $|A| = p$, $|B| = q$, and $A \leqslant_1 B$; note that for the implication 6.5.1($\Rightarrow$) to hold this requires a proof that $A' =_1 A \leqslant_1 B =_1 B' \;\Rightarrow\; A' \leqslant_1 B'$.

Theorem 6.6 (Cantor-Bernstein) *1.* $A \leqslant_1 B \leqslant_1 A \;\Rightarrow\; A =_1 B$,

2. for all cardinals p, q: $p \leqslant q \wedge q \leqslant p \;\Rightarrow\; p = q$.

Proof. 1. Assume that $f : A \to B$ and $g : B \to A$ are injections. Define $H : \wp(A) \to \wp(A)$ by $H(X) =_{\text{def}} A - g[B - f[X]]$. Clearly, H is monotone. Let X be a fixed point. Now, $(f|X) \cup (g|(B - f[X]))^{-1}$ is a bijection from A to B. □

Thus, the ordering relation of cardinals is a partial ordering. AC is equivalent with the statement that this ordering is total.

Definition 6.7 Define addition, multiplication and exponentiation for cardinals such that:

1. $|A| + |B| = |A \cup B|$, *provided* $A \cap B = \emptyset$,
2. $|A| \cdot |B| = |A \times B|$,
3. $|A|^{|B|} = |A^B|$ (N.B.: $A^B =_{\text{def}} \{f \mid f : B \to A\}$).

I.e., (e.g., in the case of 1): for cardinals p, q, r, $p + q = r$ by definition means that sets A, B exist s.t. $|A| = p$, $|B| = q$ and $|A \times \{0\} \cup B \times \{1\}| = r$; and requires a proof that $A =_1 A', B =_1 B' \;\Rightarrow\; A \times \{0\} \cup B \times \{1\} =_1 A' \times \{0\} \cup B' \times \{1\}$.

Warning. The notation A^B is ambiguous (if A and B are ordinals and/or cardinals). Cf. Definition 4.31 p. 36 and the warning given there. The context should tell what is meant: ordinal power, cardinal power, or function set. Note that, except on natural numbers, there is no connection between the first two notions. E.g., if α and β are countably infinite ordinals, then the ordinal power α^β is countable, whereas the function set has the (uncountable) power of the set of reals.

In the following, p, q, r,... denote cardinals.

Lemma 6.8

1. $p + q = q + p$; $p + (q + r) = (p + q) + r$,

2. $p \cdot q = q \cdot p$; $p \cdot (q \cdot r) = (p \cdot q) \cdot r$,

3. $p \cdot (q + r) = p \cdot q + p \cdot r$,

4. $(p^q)^r = p^{q \cdot r}$; $p^q \cdot p^r = p^{q+r}$; $(p \cdot q)^r = p^r \cdot q^r$,

5. *(Cantor's Theorem)* $|A| < 2^{|A|} = |\wp(A)|$,

6. (AC) *if* $p, q > 0$ *are not both finite, then* $p + q = p \cdot q = \max\{p, q\}$.

Proof. 1–4 are straightforward and 5 is well-known; 6 follows from Corollary 4.35. □

Lemma 6.9 *If* $p \leqslant q$*, then* $p^q = 2^q$.

Proof. $p^q \leqslant (2^p)^q = 2^{pq} = 2^q$. □

In the presence of the Axiom of Choice, we can also introduce *infinite* sums and products:

Definition 6.10

1. $\sum_{i \in I} |A_i| = |\bigcup_{i \in I} A_i|$ *povided* the A_i are pairwise disjoint,

2. $\prod_{i \in I} |A_i| = |\prod_{i \in I} A_i|$. □

N.B.: $\prod_{i \in I} A_i = \{f \mid \mathrm{Dom}(f) = I \wedge \forall i \in I\, f(i) \in A_i\}$.

The Axiom of Choice is needed here to be able to calculate $\sum_{i \in I} p_i$ and $\prod_{i \in I} p_i$: for this, we need a set A_i such that $|A_i| = p_i$ for each $i \in I$; furthermore, we do not want the evaluation of sums and products to depend on the choice of these sets, and this also requires AC.

The expected rules hold:

Lemma 6.11 *If for all* $i \in I$*:* $p_i = p$*, then* $\sum_{i \in I} p_i = p \cdot |I|$ *and* $\prod_{i \in I} p_i = p^{|I|}$.

Lemma 6.12 $p^{\sum_i q_i} = \prod_i p^{q_i}$. □

Lemma 6.13 1. $\sum_i (p \cdot q_i) = p \cdot \sum_i q_i$*,* $\sum_i (p_i + q_i) = \sum_i p_i + \sum_i q_i$*;*

2. $\prod_i p_i^q = (\prod_i p_i)^q$*,* $\prod_i (p_i \cdot q_i) = \prod_i p_i \cdot \prod_i q_i$.

Lemma 6.14 *If* $\forall i \in I (p_i \geqslant 1)$*, then* $\sum_{i \in I} p_i = \max(|I|, \sup_{i \in I} p_i)$.

Proof. (i) $\sum_{i \in I} p_i \leqslant \sum_{i \in I} \sup_{i \in I} p_i = |I| \cdot \sup_{i \in I} p_i = \max(|I|, \sup_{i \in I} p_i)$.

(ii) $|I| = \sum_{i \in I} \{i\} \leqslant \sum_{i \in I} p_i$; also, for all $i \in I$, $p_i \leqslant \sum_{i \in I} p_i$; hence $\sup_{i \in I} p_i \leqslant \sum_{i \in I} p_i$, and $\max(|I|, \sup_{i \in I} p_i) \leqslant \sum_{i \in I} p_i$. □

For all these operations, the "expected" monotonicity-laws hold relative to $\leqslant$, *but never w.r.t.* $<$. The main strict inequality is the following.

Theorem 6.15 (König-Zermelo) (AC) *If* $\forall i \in I\ (p_i < q_i)$*, then* $\sum_{i \in I} p_i < \prod_{i \in I} q_i$.

This Theorem has the cardinal-free equivalent: $\forall i \in I (A_i <_1 B_i) \Rightarrow \bigcup_{i \in I} A_i <_1 \prod_{i \in I} B_i$. With all A_i empty, this amounts to AC. For $A_i = \{i\}$ and all $B_i = 2 = \{0, 1\}$ — so, $\prod_{i \in I} B_i = 2^I =_1 \wp(I)$ — this amounts to Cantor's theorem. Thus, a proof will need both AC and a diagonal argument.

Proof. The cardinal-free form: suppose that $f : \bigcup_i A_i \to \prod_i B_i$. By AC, choose $h(i) \in B_i - \{f(a)(i) \mid a \in A_i\}$ (for a given i, the mapping $a \mapsto f(a)(i)$ cannot be a surjection from A_i onto B_i). Then $h \in \prod_i B_i - \mathrm{Ran}(f)$. □

The set of real numbers is denoted by $\mathbb{R}$. It is not difficult to see, that $\mathbb{R} =_1 \wp(\omega)$ $(=_1 2^\omega)$; i.e.: $|\mathbb{R}| = 2^{\aleph_0}$. It follows, that $|\mathbb{R}^\omega =_1 \mathbb{R}$: $|\mathbb{R}^\omega| = |\mathbb{R}|^{|\omega|} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0^2} = 2^{\aleph_0} = |\mathbb{R}|$.

Corollary 6.16 $\mathbb{R}$ *is not a countable union of sets* $<_1 \mathbb{R}$.

Proof. Assume that $\mathbb{R} = \bigcup_{n\in\omega} A_n$, where $A_n <_1 \mathbb{R}$. Then $\mathbb{R} = \bigcup_n A_n <_1 \prod_n \mathbb{R} = \mathbb{R}^\omega =_1 \mathbb{R}$, a contradiction. □

Since 1963 it is known that, in a precise sense, this corollary is all that is provable in ZFC about the cardinality of $\mathbb{R}$. For instance, as $\aleph_{123}$ is not a countable sum of smaller alpephs (i.e.: has uncountable *cofinality*, cf. Definition 6.17), ZFC cannot prove $|\mathbb{R}| \neq \aleph_{123}$; i.e., we *might have* $|\mathbb{R}| = \aleph_{123}$, as far as ZFC is concerned. Something similar holds for arbitrarily large alephs.

Exercises

109 ♣ Prove Lemma 6.8.

110 ♣ Prove Lemmas 6.11–6.13.

111 ♣ (AC) Suppose that p, q, r, s are cardinals such that $p < q$ and $r < s$. Show that $p + r < q + s$.

6.3 Cofinality and Regularity (cardinal version)

The notions of *cofinality* and *regularity* come in a *cardinal* (Definition 6.17) and in an *ordinal* (Definitions 6.28 and 6.33) version. Lemma 6.35 shows these versions match for initials and their cardinals.

The cardinal version is treated in this section. This suffices for much of the theory, but if you want to determine the cofinality of specific cardinals, the information in Section 6.4 may be useful.

Definition 6.17 Let p be a cardinal.

1. $\mathrm{cf}(p)$ is the least cardinal q such that p can be written as a sum of q-many smaller cardinals; i.e.: such that a set I and cardinals p_i $(i \in I)$ exist for which $|I| = q$, all p_i are $< p$, and $p = \sum_{i\in I} p_i$.
2. p is *regular* if $\mathrm{cf}(p) = p$ and *singular* if $\mathrm{cf}(p) < p$.

In case you want to avoid Definitions 6.28 and 6.33, the above is extended to initial numbers as follows. (Cf. Lemma 6.35.)

3. An initial ω_α is *regular* resp., *singular* if its cardinal $\aleph_\alpha$ is.
4. $\mathrm{cf}(\omega_\alpha)$ is the initial for which $|\mathrm{cf}(\omega_\alpha)| = \mathrm{cf}(\aleph_\alpha)$. □

This definition uses infinite sums and the fact that cardinals are well-ordered, and hence presupposes AC. However, there is a cofinality-definition for *alephs* that does not presuppose AC: $\mathrm{cf}(\aleph_\alpha)$ is the least aleph q such that ω_α can be partitioned into q pieces of cardinality $< \aleph_\alpha$ each. (Note that $\aleph_\alpha$ itself is such an aleph.)

Lemma 6.18 *If the cardinal p is infinite, then*

1. $\mathrm{cf}(p)$ *is infinite,*
2. $\mathrm{cf}(p) \leqslant p$*,*
3. $\mathrm{cf}(p)$ *is regular.*

Proof. 1. Obvious.
2. If $p = |I|$, then $p = \sum_{i \in I} |\{i\}| = \sum_{i \in I} 1$.
3. Assume that $p = \sum_{i \in I} p_i$, $|I| = \mathrm{cf}(p)$, $p_i < p$, and $I = \bigcup_{j \in J} I_j$, $|J|, |I_j| < |I|$. Then $p = \sum_{j \in J} \left(\sum_{i \in I_j} p_i \right)$ and (by definition of $\mathrm{cf}(p)$) for all j, $\sum_{i \in I_j} p_i < p$, contradiction. □

Lemma 6.19 *1.* $\mathrm{cf}(1)$ *is undefined,* $\mathrm{cf}(2) = \mathrm{cf}(3) = \mathrm{cf}(4) = \ldots = 2$*,*

2. $\aleph_0$ *is regular,*
3. $\aleph_{\alpha+1}$ *is regular.*

Proof. 2. Any finite union of finite sets is finite.
3. If $\aleph_{\alpha+1} = \sum_{i \in I} p_i$ with $|I|$ and all p_i smaller than $\aleph_{\alpha+1}$, then these cardinals are all $\leqslant \aleph_\alpha$, and $\aleph_{\alpha+1} = \sum_{i \in I} p_i \leqslant \sum_{i \in I} \aleph_\alpha = |I| \cdot \aleph_\alpha \leqslant \aleph_\alpha^2 = \aleph_\alpha$, a contradiction. □

What about the cofinality of limit alephs (i.e., $\aleph_\alpha$ where α is a limit)?

Definition 6.20 Regular limit alephs are called (*weakly*) *inaccessible*.

Limit alephs that "you can point at" (cf. Exercise 112) always turn out to be singular. In fact, ZFC (if consistent) is unable to prove that weak inaccessibles exist:

Theorem 6.21 *If ZFC is consistent, then so is ZFC + "no inaccessibles exist".*

However, by Gödel's 2nd incompletability theorem, there is no (sensible) consistency proof for ZFC + "some inaccessible exists", even assuming the consistency of ZFC.

Lemma 6.22 (AC)

1. $\aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)} > \aleph_\alpha$*,*
2. $\mathrm{cf}(2^{\aleph_\alpha}) > \aleph_\alpha$*.*

   (For $\alpha = 0$, this is the content of 6.16.)

The following lemma is needed here and there in the exercises below. As an example, cf. the proof of 6.37 (p. 50) where it is used.

Lemma 6.23 *If κ and λ are initials such that $\lambda < \mathrm{cf}(\kappa)$, then $\kappa^\lambda = \bigcup_{\alpha < \kappa} \alpha^\lambda$.*

Proof. Assume that $f : \lambda \to \kappa$ is such that $f \notin \bigcup_{\alpha < \kappa} \alpha^\lambda$. Then $\kappa = \bigcup_{\xi < \lambda} f(\xi) = \bigcup_{\xi < \lambda} (f(\xi) - \bigcup_{\delta < \xi} f(\delta))$ and $|\kappa| = \sum_{\xi < \lambda} |f(\xi) - \bigcup_{\delta < \xi} f(\delta)|$, contradicting $\lambda < \mathrm{cf}(\kappa)$. □

Exercises

112 ♣ Determine the cofinalities of $\aleph_\omega$, $\aleph_{\omega+\omega}$, $\aleph_{\omega_\omega}$, and $\aleph_{\omega_1}$.

113 ♣

1. Give a direct proof, using AC, but not using Lemma 6.19, that a countable union of countable sets is countable.

   In particular, ω_1 is not a countable union of countable sets. (It is known that this is unprovable without AC).

2. Show *without using AC* that ω_2 is not a countable union of countable sets.

114 ♣ Prove Lemma 6.22.
Hint. Use Theorem 6.15.

115 ♣ Suppose that α is a limit and that $\aleph_\beta < \mathrm{cf}(\aleph_\alpha)$. Show that $\aleph_\alpha^{\aleph_\beta} = \sum_{\gamma<\alpha} \aleph_\gamma^{\aleph_\beta}$.

116 ♣ (Hausdorff) Prove that $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}$.
Hint. Distinguish as to whether $\aleph_\beta$ is $<$ or $\geqslant$ than $\aleph_{\alpha+1}$.

117 ♣ Show that for all $n, m \in \omega$: $\aleph_n^{\aleph_m} = \aleph_n \cdot 2^{\aleph_m}$.

118 ♣ Show: $\aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1}$.

119 ♣ (Bukovsky; Hechler) Assume that $\aleph_\alpha$ is singular and that $\beta < \alpha$ exists such that $\beta \leqslant \gamma < \alpha \ \Rightarrow\ 2^{\aleph_\gamma} = 2^{\aleph_\beta}$. Show that $2^{\aleph_\alpha} = 2^{\aleph_\beta}$.

Definition 6.24 A cardinal p is a *strong limit* if for all cardinals q, $q < p \ \Rightarrow\ 2^q < p$. An uncountable regular strong limit cardinal (and the corresponding initial ordinal) is called *strongly inaccessible.*

Obviously, if GCH holds, then the notions of *weak inaccessible* and *strong inaccessible* coincide.

120 ♣ Show: if κ is a strongly inaccesible initial number, then

1. $\beta < \kappa \ \Rightarrow\ \mathrm{V}_\beta <_1 \kappa$,

2. $\mathrm{V}_\kappa =_1 \kappa$,

3. $(\mathrm{V}_\kappa, \in)$ satisfies all ZFC Axioms,

4. if κ is the *least* strong inaccessible, then $(\mathrm{V}_\kappa, \in) \models$"there is no strong inaccessible".

(Because of the definition of strong inaccessible, part 4 is tedious; parts 1–3 are straightforward.)

Here is a proof that ZFC + "no strong inaccessible exists" is consistent (provided ZF is consistent). For the logical background of this proof, see Section 7.2 (p. 52).

Suppose this theory is *not* consistent. That means: you can prove existence of an inaccessible in ZFC. By Exercise 120.3/4, you can construct an inner model of ZFC that has no inaccessible, contradicting the assumed existence of a proof that inaccessibles exist. □

Since GCH is consistent, and, under it, strong and weak inaccessibles are the same, non-existence of weak inaccessibles is consistent as well.

Let ZFCI be the theory: ZFC+"an inaccessible exists".

Here is a proof that you *cannot* hope to prove ZFCI consistent by assuming only consistency of ZFC:
Suppose you had such a proof, showing that (ZFC consistent $\Rightarrow$ ZFCI consistent).

Whatever its details, you should at least be able to formalize it in the extremely strong theory ZFCI. (In fact, trustworthy consistency proofs should only deal with concrete objects: proofs and their combinatorial properties, and therefore they usually will be formalizable already in ZF without Infinity Axiom.)

By Exercise 120, ZFCI proves that ZFC has a model, and hence ZFCI proves that ZFC is consistent.

Since your hypothetical proof of the implication (ZFC consistent $\Rightarrow$ ZFCI consistent) has been formalized in ZFCI, it follows (by Modus Ponens) that ZFCI proves ZFCI consistent.

Now Gödel's second incompletability theorem says that (under suitable conditions that are satisfied here) every theory that proves itself consistent must be, in fact, inconsistent. Therefore, ZFCI is inconsistent.

But then, your hypothetical proof shows that, by contraposition, also ZFC must be inconsistent! □

Note the asymmetric behavior of ZFC with regard to the statement "an inaccessible exists" and its negation: the latter one is easily shown to be consistent relative to ZF, whereas consistency of ZF alone is insufficient to prove the former one to be consistent. Compare this to e.g., AC and CH: these, as well as their negations, have been shown consistent relative to ZF alone.

6.4 Cofinality and Regularity (ordinal version)

Here comes the ordinal version of these notions.

Definition 6.25 A subset $B \subset A$ is *cofinal* in the linear ordering $(A, <)$ if $\forall a \in A\, \exists b \in B\, (a \leqslant b)$. A non-cofinal set is called *bounded.* □

Lemma 6.26 *If f maps the ordinal α onto a cofinal subset of the linear ordering $(A, \prec)$, then $X \subset \alpha$ exists such that $f|X$ is an order-preserving map between X and a cofinal subset of A.*

Proof. Put $X =_{\text{def}} \{\xi < \alpha \mid \forall \delta < \xi\, [f(\delta) \prec f(\xi)]\}$. Clearly, $f|X$ is order-preserving (if $\xi, \delta \in X$ and $\delta < \xi$, then $f(\delta) < f(\xi)$). And if $a \in A$ and ξ is the least ordinal s.t. $a \leqslant f(\xi)$, then $\xi \in X$, and so X is cofinal. □

Corollary 6.27 (AC) *Every linear ordering has a well-ordered cofinal set.*

Proof. Apply Lemma 6.26 to any surjection from an ordinal to the linear ordering. □

Definition 6.28

1. (AC) Let $(A, <)$ be a linear ordering. The *cofinality* of $(A, <)$, $\mathrm{cf}(A, <)$, is the least ordinal that is the type of a cofinal set of $(A, <)$.
2. $\mathrm{cf}(\alpha) =_{\mathrm{def}} \mathrm{cf}(\alpha, <)$. □

Examples.

1. $\mathrm{cf}(0) = 0$,
2. $\mathrm{cf}(\alpha) = 1$ iff α is a successor,
3. $\mathrm{cf}(\alpha)$ is a limit iff α is a limit,
4. $\mathrm{cf}(\omega) = \mathrm{cf}(\omega + \omega) = \mathrm{cf}(\omega^\omega) = \mathrm{cf}(\omega_\omega) = \omega$,
5. every countable limit ordinal has cofinality ω,
6. $\mathrm{cf}(\omega_1) = \omega_1$ (cf. Lemma 6.34).

Lemma 6.29 *If f maps α onto a cofinal subset of β, then* $\mathrm{cf}(\beta) \leqslant \alpha$.

Proof. By Lemma 6.26, let $X \subset \alpha$ be such that $f|X$ maps X order-preserving onto the cofinal subset $f[X]$ of β. Then (by Exercise 61 p. 28): $\mathrm{cf}(\beta) \leqslant \mathrm{type}(f[X]) = \mathrm{type}(X) \leqslant \alpha$.
□

By Definition 6.28, $\mathrm{cf}(\beta)$ is the least ordinal that *order-preservingly* can be mapped onto a cofinal subset of β. By the previous Lemma, you can forget about the *order-preserving* requirement here:

Corollary 6.30 $\mathrm{cf}(\beta)$ *is the least ordinal that can be mapped onto a cofinal subset of β.*
□

Proof. Suppose that α is the least ordinal that can be mapped onto a cofinal subset of β. Then, since $\mathrm{cf}(\beta)$ can be so mapped, $\alpha \leqslant \mathrm{cf}(\beta)$. And by Lemma 6.29, $\mathrm{cf}(\beta) \leqslant \alpha$. □

The following corollary is the main instrument relating cofinality and cardinal exponentiation.

Corollary 6.31 *If $\alpha < \mathrm{cf}(\beta)$, then β^α $(= \{f \mid f : \alpha \to \beta\})$ $= \bigcup_{\xi<\beta} \xi^\alpha$.*

Corollary 6.32 *If α is a limit, then $\mathrm{cf}(\alpha)$ is an initial number that is $\leqslant \alpha$.* □

Definition 6.33 A limit ordinal α is *regular* if $\mathrm{cf}(\alpha) = \alpha$; if $\mathrm{cf}(\alpha) < \alpha$, then α is called *singular*. □

By Exercise 122.3, not every initial is regular. E.g., if α is a limit such that $\alpha < \omega_\alpha$, then $\mathrm{cf}(\omega_\alpha) = \mathrm{cf}(\alpha) \leqslant \alpha < \omega_\alpha$. For instance, $\mathrm{cf}(\omega_\omega) = \omega$; by Exercise 122.3 and Lemma 6.34, $\mathrm{cf}(\omega_{\omega_1}) = \mathrm{cf}(\omega_1) = \omega_1 < \omega_{\omega_1}$.

The least regular initial is ω. All successor initials are regular:

Lemma 6.34 (AC) $\omega_{\alpha+1}$ *is regular.*

Proof. Assume that $\omega_{\alpha+1}$ has a cofinal subset B such that $B <_1 \omega_{\alpha+1}$. For every $\beta \in B$, choose an injection : $\beta \to \omega_\alpha$. Then $\omega_{\alpha+1} = \bigcup B \leqslant_1 \bigcup_{\beta\in B} \beta \times \{\beta\} \leqslant_1 \omega_\alpha \times B \leqslant_1 \omega_\alpha \times \omega_\alpha$ $=_1 \omega_\alpha$. □

Note that this proof shows, in particular, the familiar fact that a countable union of countable sets is countable. But, even this needs AC.

The following lemma presents the close connection between ordinal and cardinal notions.

Lemma 6.35 *1.* $\mathrm{cf}(\aleph_\alpha) = |\mathrm{cf}(\omega_\alpha)|$, *and hence*

2. $\aleph_\alpha$ *is regular iff* ω_α *is regular.*

Proof. Using the AC-free definition, this can be shown without AC. Here is a proof using AC: Assume that $\beta = \mathrm{cf}(\omega_\alpha)$ and that $f : \beta \to \omega_\alpha$ is an order-preserving map onto a cofinal subset of ω_α. Then $\omega_\alpha = \bigcup_{\xi<\beta} f(\xi) = \bigcup_{\xi<\beta}[f(\xi) - \bigcup_{\xi'<\xi} f(\xi')]$ is a partition of ω_α into β pieces of power $< \aleph_\alpha$ and hence we have that $\mathrm{cf}(\aleph_\alpha) \leqslant |\mathrm{cf}(\omega_\alpha)|$.
Conversely, assume that $|\beta| = \mathrm{cf}(\aleph_\alpha)$.
First, suppose that $\mathrm{cf}(\aleph_\alpha) < \aleph_\alpha$. Say, $\aleph_\alpha = \sum_{\xi<\beta} |\beta_\xi|$, where $|\beta_\xi| < \aleph_\alpha$. Then $\{\beta_\xi \mid \xi < \beta\}$ is cofinal in ω_α. (For, if $\{\beta_\xi \mid \xi < \beta\} \subset \gamma < \omega_\alpha$, then $\aleph_\alpha = \sum_{\xi<\beta} |\beta_\xi| \leqslant \sum_{\xi<\beta} |\gamma| = |\beta| \cdot |\gamma|$ $= \max\{|\beta|, |\gamma|\} < \aleph_\alpha$.) Hence, $|\mathrm{cf}(\omega_\alpha)| \leqslant |\beta| = \mathrm{cf}(\aleph_\alpha)$.
Finally, if $\mathrm{cf}(\aleph_\alpha) = \aleph_\alpha$, then also $\mathrm{cf}(\omega_\alpha) = \omega_\alpha$: otherwise we had, by the previous argument: $\mathrm{cf}(\aleph_\alpha) \leqslant |\mathrm{cf}(\omega_\alpha)| < |\omega_\alpha| = \aleph_\alpha$. □

Exercises

121 ♣ Show: $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$.

122 ♣ Show:

1. if $X \subset \alpha$ is cofinal in α, then α has a cofinal subset Y of type $\mathrm{cf}(\alpha)$ such that $Y \subset X$,
2. if α and β have cofinal subsets of the same type, then $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$,
3. if α is a limit, then $\mathrm{cf}(\omega_\alpha) = \mathrm{cf}(\alpha)$.

123 ♣ Show:

1. if α is a limit such that ω_α is regular, then $\omega_\alpha = \alpha$,
2. if α is the least ordinal such that $\omega_\alpha = \alpha$, then $\mathrm{cf}(\alpha) = \omega$,
3. if ω_α is weakly inaccessible, then $\{\beta < \alpha \mid \omega_\beta = \beta\} =_1 \alpha$.

6.5 Continuum Hypothesis

The *Continuum Problem* asks to determine the cardinality of the set of reals $\mathbb{R}$. If AC holds, this is particularly pressing: calculate α such that $\aleph_\alpha = |\mathbb{R}|$ $(= 2^{\aleph_0})$. All that we know so far is that (Cantor) $2^{\aleph_0} \neq \aleph_0$. In fact (König-Zermelo): $\mathrm{cf}(2^{\aleph_0}) \neq \aleph_0$. This excludes cardinals such as $\aleph_0, \aleph_\omega, \aleph_{\omega^\omega}, \ldots$ as values of $2^{\aleph_0}$.

Cantor's *Continuum Hypothesis*, CH, asserts that $|\mathbb{R}|$ $(= 2^{\aleph_0}) = \aleph_1$. The *Generalized Continuum Hypothesis*, GCH, is the statement: for *every* ordinal α: $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

Gödel proved in 1938 (using the inner model of *constructible sets*, cf. Chapter 7) the consistency of ZFC plus GCH assuming consistency of ZF. Cohen showed in 1963 (by means of his *forcing method*) that there are lots of (consistent) alternatives to CH. In fact, Easton showed in 1964 that, roughly, for all *regular* cardinals $\aleph_\alpha$ simultaneously, *every* cardinal value for $2^{\aleph_\alpha}$ is consistent, provided the following (obviously, by 6.22.2, necessary) requirements be satisfied:

- $\alpha < \beta \;\Rightarrow\; 2^{\aleph_\alpha} \leqslant 2^{\aleph_\beta}$, and
- $\mathrm{cf}(2^{\aleph_\alpha}) > \aleph_\alpha$.

The restriction to powers of *regular* cardinals appeared to be essential. For instance, Exercise 119 shows a case where a singular power depends on powers of smaller regulars. For another one, there is a remarkable theorem, a genuine new result in the field of cardinal arithmetic since about 70 years, of which the following is but an instance:

Theorem 6.36 (Silver 1974) *If for all $\xi < \omega_1$, $2^{\aleph_\xi} = \aleph_{\xi+1}$, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$.*

That is, GCH cannot first fail at $\aleph_{\omega_1}$.

Finally, Jensen proved that *if $0^\sharp$ does not exist*, then all powers of singular cardinals can be calculated from the values of the regular ones. Here, $0^\sharp$ is a subset of ω with specific properties, the existence of which only follows from certain large cardinal hypotheses: a strange connection between such a relatively simple object as a set of natural numbers and the values of cardinal powers high up.

The following lemma shows that GCH trivializes exponentiation of cardinals. In it, p^+ denotes the least cardinal $> p$ (that is, $\aleph_\alpha^+ = \aleph_{\alpha+1}$).

Lemma 6.37 (AC, GCH) *If $p, q \geqslant \aleph_0$, then*

$$q^p = \begin{cases} q & \text{if } p < \mathrm{cf}(q) \\ q^+ & \text{if } \quad \mathrm{cf}(q) \leqslant p \leqslant q \\ p^+ & \text{if } \qquad\qquad q \leqslant p. \end{cases}$$

Proof. If $q \leqslant p$, then $2^p \leqslant q^p \leqslant (2^q)^p = 2^p = p^+$.
If $\mathrm{cf}(q) \leqslant p \leqslant q$, then (by Lemma 6.22.1) $q < q^{\mathrm{cf}(q)} \leqslant q^p \leqslant (2^q)^p = 2^q = q^+$.
Finally, assume that α and β are initials of power p resp. q. If $p < \mathrm{cf}(q)$ then $\alpha < \mathrm{cf}(\beta)$ and hence (Corollary 6.31 p. 48 or Lemma 6.23 p. 45) $\beta^\alpha = \bigcup_{\xi<\beta} \xi^\alpha$. Thus,
$q \leqslant q^p = |\beta^\alpha| = |\bigcup_{\xi<\beta} \xi^\alpha| \leqslant \sum_{\xi<\beta} |\xi^\alpha| \leqslant \sum_{\xi<\beta} 2^{|\xi||\alpha|} = \sum_{\xi<\beta} \max(|\xi|^+, |\alpha|^+) \leqslant \sum_{\xi<\beta} |\beta| = |\beta| \cdot |\beta| = |\beta| = q.$ □

Chapter 7

Consistency of AC and GCH

7.1 Preliminaries

Gödel, in 1935–8, proved that ZFC+GCH (=ZF+AC+GCH) is consistent (doesn't prove a contradiction), *provided* ZF is consistent. For his proof, he constructed the first non-trivial "inner model" for ZF: the class **L** of *constructible* sets (Definition 7.19 p. 61). Gödel published his proof first, in a couple of very short papers, for ZF set theory, and later, in a lengthier monograph, for NBG set theory. The presentation below is the nowadays usual one for ZF.

ZF set theory is a first-order theory over the vocabulary whose sole non-logical constant is the binary relation symbol $\in$. Recall the list of ZF axioms in this formalism:

1. Extensionality

$$\forall a\, \forall b\, [\forall x(x \in a \leftrightarrow x \in b) \;\rightarrow\; a = b].$$

2. Separation or **Aussonderung** (Φ any formula that doesn't contain b freely)

$$\forall a\, \exists b\, \forall x\, [x \in b \;\leftrightarrow\; x \in a \;\wedge\; \Phi].$$

3. Pairing

$$\forall a\, \forall b\, \exists c\, \forall x\, (x \in c \;\leftrightarrow\; x = a \;\vee\; x = b).$$

4. Sumset

$$\forall a\, \exists b\, \forall x\, [x \in b \;\leftrightarrow\; \exists y(x \in y \;\wedge\; y \in a)].$$

5. Powerset

$$\forall a\, \exists b\, \forall y\, [y \in b \;\leftrightarrow\; \forall x(x \in y \rightarrow x \in a)].$$

6. Substitution or **Replacement** (Ψ any formula that doesn't contain b freely)

$$\forall a\, [\forall x \in a\, \exists! y\, \Psi \;\rightarrow\; \exists b \forall y(y \in b \leftrightarrow \exists x \in a \Psi)]\,.$$

7. Infinity

$$\exists a\, [\, \exists x \in a\, \forall y\, (y \notin x) \;\wedge\; \forall x \in a\, \exists y \in a\, \forall z\, (z \in y \;\leftrightarrow\; z \in x \;\vee\; z = x)].$$

8. Foundation

$$\forall a\, [\, \exists x\, (x \in a) \;\rightarrow\; \exists x \in a\, \forall y \in a\, (y \notin x)].$$

The *Axiom of Choice* (AC) says that every set has a choice function. This is equivalent to the Well-ordering Theorem: for every set there is a relation that well-orders it.

The *Continuum Hypothesis* (CH) says that the set of reals —which has cardinality $2^{\aleph_0}$— has cardinality $\aleph_1$. The *Generalized Continuum Hypothesis* (GCH) says that if A is any infinite set, then the cardinality of $\wp(A)$ is the least one that is bigger than the one of A. If AC holds, then infinite cardinalities are alephs; and if $|A| = \aleph_\alpha$, then $|\wp(A)| = 2^{\aleph_\alpha}$ and the least cardinal $> \aleph_\alpha$ is $\aleph_{\alpha+1}$. Thus, GCH takes the simple form: $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

The first main result of Gödel on the constructible universe $\mathbf{L}$ is the following.

Theorem 7.1 *The class* $\mathbf{L}$ *is an "inner model" of* $\mathrm{ZF} + \mathbf{V} = \mathbf{L}$ *in* ZF.

That is, it is provable in ZF that all ZF-axioms and the statement $\mathbf{V} = \mathbf{L}$ hold *in* $\mathbf{L}$.

The statement $\mathbf{V} = \mathbf{L}$, expressing that every set is constructible, is called the *Axiom of Constructibility.*

As a consequence, we now have the following relative consistency result:

Corollary 7.2 *If* ZF *is consistent, then so is* $\mathrm{ZF} + \mathbf{V} = \mathbf{L}$.

Proof. See next section. □

The following is Gödel's second main result ($\vdash$ means "proves").

Theorem 7.3 $\mathrm{ZF} + \mathbf{V} = \mathbf{L} \vdash \mathrm{AC},\ \mathrm{GCH}$.

Corollary 7.4 *If* ZF *is consistent, then so is* $\mathrm{ZFC} + \mathrm{GCH}$.

Proof. Immediate from Corollary 7.2 and Theorem 7.3. □

7.2 Logical Background: Interpretation, Inner Model

This section explains the logical background of relative consistency results using "inner models"; in particular, it explains the meaning of Theorem 7.1 and proves Corollary 7.2.

The first thing to be clarified is: what do we mean by saying that (ZF proves that) a certain set-theoretic sentence *holds in* a certain class $\mathbf{U}$?

ZF axioms and theorems (at least, when written in the ZF-language not using defined symbols) are intended as assertions about the system $\mathcal{V} = (\mathbf{V}, \in)$ (where $\mathbf{V}$ is the collection of all sets and $\in$ the relation of membership), reading the formalism in the following way:

1. quantifiers $\forall x$ and $\exists y$ are shorthands for: "for all sets x", resp. "for some set y",
2. atomic statements $x \in y$ are taken to mean: "x is an element of y";

and, of course, interpretations of atomic statements $x = y$ and connectives are independent of the "context" $\mathcal{V}$.

Now, such formal statements can be similarly interpreted relative to arbitrary systems $\mathcal{U} = (\mathbf{U}, \varepsilon)$ (cf. the fact that logical sentences can be interpreted in different models). Simply, read

1' quantifiers $\forall x$ and $\exists y$ as: "for all things x in $\mathbf{U}$", resp., "for some thing y in $\mathbf{U}$";

2' atomic statements $x \in y$ as: $x \mathrel{\varepsilon} y$.

(Interpretations of atomic statements $x = y$ and connectives are still independent of the "context" $\mathcal{U}$.)

Definition 7.5 Interpretation, Inner model. A (definition of a) system $\mathcal{U} = (\mathbf{U}, \varepsilon)$ in ZF, where ZF proves that $\mathbf{U} \neq \emptyset$, is the same as what is usually called a (relative) *interpretation* (of the ZF language in the ZF language). E.g., forcing interpretations are of this kind.

In the special case that $\varepsilon = \in$, the interpretation of atomic statements $x \in y$ doesn't change, and only quantifiers have to be re-interpreted. In such a case, the interpretation, which now is completely determined by $\mathbf{U}$, is called an *inner model*.

The inner model $\mathbf{U}$ is an inner model *of the collection of sentences* Σ *in ZF* in case ZF proves that Φ holds in $\mathbf{U}$ for every $\Phi \in \Sigma$.

N.B.: A sentence Φ is allowed to have free variables. For instance, the formulas used in instances of the ZF axiom schemas of Separation and Substitution may contain such free variables. These should be thought of as *universally quantified*. Thus: to say that such a Φ holds in $\mathbf{U}$ means: with values for these variables that come from the inner model $\mathbf{U}$.

Note that the class $\mathbf{U}$ need not be a set. Thus, an inner model often is not a model (the universe of which must be a set); it is an interpretation that is a syntactic pendant of a model. The relation with the notion of a model is the following: an inner model (defined in ZF) determines, in a uniform way, a submodel of any given model (of ZF). Moreover, if, e.g., ZF proves that Φ holds in the inner model $\mathbf{U}$ (i.e.: $\mathbf{U}$ is an inner model of Φ in ZF), then Φ is true in every $\mathbf{U}$-determined submodel of a model of ZF: see the model-theoretic proof of Lemma 7.6 given below. More on models vs. inner models in Remark 7.48 (p. 71).

The above explains how one can re-interpret statements w.r.t. a given class $\mathbf{U}$.

Next, it is important to note that such re-interpretations can be described in the following formal way. When Φ is a sentence in the ZF-language, let $\Phi^{\mathbf{U}}$ be the sentence in which this reinterpretation has been "carried out"; i.e.: all quantifiers $\forall x$ and $\exists y$ have been replaced by $\forall x \in \mathbf{U}$, resp. $\exists y \in \mathbf{U}$. ($\Phi^{\mathbf{U}}$ is called the *relativization* of Φ w.r.t. $\mathbf{U}$.) Now what is meant by saying that a certain sentence Φ *holds in* a system $(\mathbf{U}, \in)$ can be explained as asserting that $\Phi^{\mathbf{U}}$ holds (in $(\mathbf{V}, \in)$, that is). (Compare Lemma 7.7.) Note: $\mathbf{U}$ is a class, i.e.: it has a defining formula, and therefore $\Phi^{\mathbf{U}}$ can be considered a sentence in the ZF-language.

The main fact about such relativized sentences and deducibility is the following.

Lemma 7.6 *If* $\Sigma \vdash \Phi$*, then:* $\Sigma^{\mathbf{U}}, \mathbf{U} \neq \emptyset \vdash \Phi^{\mathbf{U}}$.

That is, if you can prove, from axioms Σ, that Φ holds, then there is a proof of $\Phi^{\mathbf{U}}$ from the $\mathbf{U}$-reinterpreted axioms $\Sigma^{\mathbf{U}}$ (together with the assumption that $\mathbf{U} \neq \emptyset$).

This is a purely logical result that has nothing to do with set theory: it holds in every first-order context. Furthermore, the derivation of $\Phi^{\mathbf{U}}$ from $\Sigma^{\mathbf{U}}$ and $\mathbf{U} \neq \emptyset$ can be obtained in a mechanical way from the one deriving Φ from Σ. As a consequence, the following proof of Corollary 7.2 is strictly "finitary". (The need for the extra hypothesis $\mathbf{U} \neq \emptyset$ is explained by the example that $\exists x(x = x)$ is —due to a logical convention that we disregard empty domains— provable from the empty set of hypotheses, but, for $\mathbf{U} = \{x \mid x \neq x\}$, the relativization $\exists x \in \mathbf{U}(x = x)$, being a contradiction, is unprovable.)

Proof of Corollary 7.2. Every proof of a contradiction in ZFC+$\mathbf{V} = \mathbf{L}$ can be transformed into a proof of a contradiction in ZF. For, suppose that ZFC+$\mathbf{V} = \mathbf{L}$ is inconsistent. An example of a contradictory statement is $\exists x(x \neq x)$. So, ZFC+$\mathbf{V} = \mathbf{L} \vdash \exists x(x \neq x)$. By Lemma 7.6, $(\mathrm{ZF} + \mathbf{V} = \mathbf{L})^{\mathbf{L}}$, $\mathbf{L} \neq \emptyset \ \vdash (\exists x(x \neq x))^{\mathbf{L}}$. But, $(\exists x(x \neq x))^{\mathbf{L}}$ is the sentence $\exists x \in \mathbf{L}(x \neq x)$, which is just another contradiction. So, $(\mathrm{ZF} + \mathbf{V} = \mathbf{L})^{\mathbf{L}} + \mathbf{L} \neq \emptyset$ is inconsistent. However, by Theorem 7.1, all axioms from $(\mathrm{ZF} + \mathbf{V} = \mathbf{L})^{\mathbf{L}}$ are provable in ZF. Moreover, we shall see that ZF proves that, e.g., $\emptyset \in \mathbf{L}$, i.e., that $\mathbf{L} \neq \emptyset$. It follows that ZF is inconsistent as well. □

Finally, here is some insight on why Lemma 7.6 is true. A *proof-theoretic* proof of Lemma 7.6 depends on the actual first-order proof system chosen (see Exercise 124). However, by Gödel's Completeness Theorem, first-order provability $\vdash$ coincides with logical consequence $\models$. Therefore, the lemma amounts to: if $\Sigma \models \Phi$, then $\Sigma^{\mathbf{U}}$, $\mathbf{U} \neq \emptyset \ \models \ \Phi^{\mathbf{U}}$.

To verify this, a second lemma is needed. Let $\mathcal{V} = (\mathbf{V}, \in)$ be a structure appropriate to the set-theoretic language; that is: $\mathbf{V}$ is a non-empty set and $\in$ is a binary relation on $\mathbf{V}$. Suppose that $\emptyset \neq \mathbf{U} \subset \mathbf{V}$. Let $\mathcal{U} = (\mathbf{U}, \in)$ be the submodel of $\mathbf{V}$ that has universe $\mathbf{U}$. (By convention, models must be non-empty.)

Lemma 7.7 *For every sentence Φ: $\mathcal{U} \models \Phi$ iff $\mathcal{V} \models \Phi^{\mathbf{U}}$.*

Proof of this can be read off from the description of $\Phi^{\mathbf{U}}$. Allowing Φ to be a formula with arguments from $\mathbf{U}$, it is straightforward to prove the equivalence using induction w.r.t. Φ.

Now, to prove Lemma 7.6, assume that $\mathcal{V} \models \Sigma^{\mathbf{U}}, \mathbf{U} \neq \emptyset$. Then by Lemma 7.7, $\mathcal{U} \models \Sigma$. Assuming that Φ follows logically from Σ, you obtain that $\mathcal{U} \models \Phi$. Therefore, again by Lemma 7.7, $\mathcal{V} \models \Phi^{\mathbf{U}}$. This completes proof that $\Sigma^{\mathbf{U}}$, $\mathbf{U} \neq \emptyset \ \models \ \Phi^{\mathbf{U}}$.

Proving relative consistency of set theories, a model theoretic argument as this one — involving set-theoretic arguments! — may appear doubtful, and therefore a proof theoretic argument is preferable; but the argument given at least shows you how things fit together, and the model-theoretic viewpoint provides a good intuition of the why and how.

Exercises

124 ♣ Prove Lemma 7.6 where $\vdash$ is (defined by) natural deduction.
Hint. First, generalize the lemma to the following statement involving free variables: if $\Sigma \vdash \Phi$ and $x_1, \ldots, x_n$ is a list of all variables occurring freely in Σ and Φ, then

$$x_0, x_1, \ldots, x_n \in \mathbf{U}, \Sigma^{\mathbf{U}} \vdash \Phi^{\mathbf{U}}.$$

(Note that the extra hypothesis $x_0 \in \mathbf{U}$ implies that $\mathbf{U} \neq \emptyset$.) Use induction w.r.t. derivations. In particular, have a look at the quantifier rules.

125 ♣ Recall (Exercise 16 p. 10 and Theorem 4.18 p. 32) that ($\mathbf{G}$ is the least class such that) $\wp(\mathbf{G}) = \mathbf{G}$. Show, in ZF *minus* Foundation:

1. If K is a class such that $\wp(K) = K$, then every ZF axiom —with the possible exception of Foundation— holds in K.

2. Foundation is true in $\mathbf{G}$.

Thus, **G** is an inner model for ZF *including* Foundation in ZF *minus* Foundation. Therefore, if the latter theory is consistent, then so is the former.

126 ♣ Put $K = \{a \mid \text{ there is no strongly inaccessible initial} \leqslant \rho(a)\}$. (Thus, $K = \mathbf{V}$ in case there is no strong inaccessible, and otherwise $K = \mathrm{V}_\lambda$, where λ is the least strongly inaccessible initial.) Show that K is an inner model of ZF together with the statement "there is no strong inaccessible" into ZF. (Again, there is the corresponding relative consistency result.)

127 ♣ Show that the following ZF axioms cannot be deduced from the others (modulo a consistency assumption):

1. Infinity,
2. Powerset,
3. Substitution (e.g., existence of $\omega + \omega$ is unprovable),
4. Sumsets.

7.3 Transitive Inner Models and Absoluteness

Note: From now on, the Foundation Axiom is assumed as part of the ZF axiomatics.

This section provides some general techniques for showing a collection to model certain ZF axioms.

In the sequel, K is an arbitrary non-empty collection.

Exercises

128 ♣ The Foundation Axiom holds in K.
Hint. Suppose that $K \cap a \neq \emptyset$. $K \cap a$ is a set (Separation). Apply Foundation to it.

129 ♣ Assume that K is transitive, that is: $x \in y \wedge y \in K \Rightarrow x \in K$. Show that the Extensionality Axiom holds in K.

ZF-formulas are built from atoms ($x \in y$, $x = y$) by means of the connectives ($\neg$, $\rightarrow$, $\leftrightarrow$, $\wedge$, $\vee$) and the quantifiers ($\forall x$, $\exists y$). Note that defined notions are not allowed.

Definition 7.8 A ZF-formula $\Phi(x_1, \ldots, x_n)$ with free variables $x_1, \ldots, x_n$ is *absolute w.r.t.* K if, for all $a_1, \ldots, a_n \in K$, it holds (in $\mathbf{V}$) that:

$$\Phi^K(a_1, \ldots, a_n) \leftrightarrow \Phi(a_1, \ldots, a_n).$$

More generally, when $K \subset W$, $\Phi(x_1, \ldots, x_n)$ is *absolute between* K *and* W if, for all $a_1, \ldots, a_n \in K$, it holds that:

$$\Phi^K(a_1, \ldots, a_n) \leftrightarrow \Phi^W(a_1, \ldots, a_n).$$

Thus, absoluteness of Φ w.r.t. K —which means that the statement $\forall a_1,\ldots,a_n \in K\,\big(\Phi^K(a_1,\ldots,a_n) \leftrightarrow \Phi(a_1,\ldots,a_n)\big)$ is true— says that Φ expresses in K what it expresses in $\mathbf{V}$, that is: what you used to think Φ was expressing.

Examples where Φ is a sentence (has no free variables):
Exercise 128 says that the Foundation Axiom is absolute w.r.t. any collection K.
Exercise 129 says that, if K is transitive, the Extensionality Axiom is absolute w.r.t. K.
The sentence $\forall x \forall y(x \in y \vee y \in x \vee x = y)$ is not absolute w.r.t. the class OR of ordinals (for, it holds in OR but is false in $\mathbf{V}$).

Warning. The notion of absoluteness (w.r.t. a some class K) makes sense only for well-defined ZF formulas (in particular, defined notions are not allowed). Note that, in the ZF-context, one often identifies equivalent formulas; and hence it may be unclear what the precise formula is that is claimed to be absolute. (An equivalence that holds in $\mathbf{V}$ may well fail to hold in the class K.) Only when it is known that the class K satisfies ZF, one may become careless in this respect.

To say that some collection is absolute is to say that its defining formula is absolute; again: it must be clear which defining formula is meant in order for the absoluteness claim to make precise sense.

Transitivity. The advantage of *transitive* inner models is that a large number of important formulas (the so-called *bounded* ones) is absolute w.r.t. them.
Note: by Exercise 67 p. 29, every class model of the Extensionality Axiom is ($\in$-$\in$-) isomorphic to a transitive class. Therefore, there is no loss of generality in restricting the discussion to *transitive* inner models.

130 ♣ Assuming Foundation, the formula that says "x is a transitive set of transitive sets" defines the collection OR of all ordinals. Show that this formula is absolute w.r.t. transitive collections. (Cf. Lemma 7.12.7.)

Notation 7.9 If the class M is defined by the formula Φ, i.e.: if $M = \{x \in \mathbf{V} \mid \Phi(x)\}$, then it is usual to write $M^K =_{\text{def}} \big\{x \in K \mid \Phi^K(x)\big\}$.

Thus, for transitive K, $\text{OR}^K = K \cap \text{OR}$.

131 ♣ Note: the formula $\forall u(u \in z \leftrightarrow u = x \vee u = y)$ expresses that $z = \{x, y\}$.

1. Show that it is absolute w.r.t. transitive classes K. (Cf. Lemma 7.12.3.)
2. (A sufficient condition under which K satisfies Pairing.)

   Assume that K is transitive and that $\forall a, b \in K(\{a, b\} \in K)$, that is: K is closed under formation of pairs. Show that the Pairing Axiom (that is: the sentence $\forall x, y \exists z \forall u(u \in z \leftrightarrow u = x \vee u = y)$) holds in K.

132 ♣ Note: the formula $\forall z\,(z \in y \leftrightarrow \exists u(u \in x \wedge z \in u))$ expresses that $y = \bigcup x$.

1. Show that it is absolute w.r.t. transitive collections. (Cf. Lemma 7.12.10.)
2. (A sufficient condition under which K satisfies the Sumset Axiom.)

   Assume that K is transitive and that $\forall a \in K(\bigcup a \in K)$, that is: K is closed under formation of sumsets. Show that the Sumset Axiom (that is: the sentence $\forall x \exists y \forall z\,(z \in y \leftrightarrow \exists u(u \in x \wedge z \in u)))$ holds in K.

Powersets behave differently from pairs and sumsets in this respect: the formula $\forall z\,(z \in y \leftrightarrow \forall u(u \in z \to u \in x))$ (expressing that $y = \wp(x)$) is in general *not* absolute. (The proof of this rests on the Downward Löwenheim-Skolem Theorem; cf. below.)

133 ♣

1. Assume that K is transitive. Show that, for $a, b \in K$:
   $[\forall z(z \in b \leftrightarrow \forall u(u \in z \to u \in a))]^K$ (i.e., $[b = \wp(a)]^K$) iff $b = K \cap \wp(a)$.
2. (A sufficient condition under which K satisfies the Powerset Axiom.)
   Assume that K is transitive and that $\forall a \in K\,(K \cap \wp(a) \in K)$. Show that the Powerset Axiom holds in K.

Definition 7.10 The class of *bounded* or *restricted* formulas of the ZF language is generated from the atomic ones by means of all connectives and the *bounded quantifiers* $\forall x \in y$ and $\exists x \in y$.

This class is also designated by Δ_0 and by Σ_0. (It is the starting point of a hierarchy of set-theoretic formulas; cf. Definition 7.13.)

Note that, in bounded formulas, the quantifier bounds must be given by variables. Therefore, formulas of the form Φ^K in general are not bounded.

The importance of bounded formulas is (i) they are absolute w.r.t. transitive collections (Lemma 7.11), and (ii) a great number of important set-theoretic notions (though not all of them) are given by bounded formulas (Lemma 7.12).

Lemma 7.11 *Bounded formulas are absolute w.r.t. transitive collections.*

Proof. Induction w.r.t. the nr. of logical symbols (connectives, bounded quantifiers) in the formula. Only the induction-step for (existential) quantification needs an argument. Thus, assume that (induction hypothesis) the formula $\Phi(x, y, z)$ is absolute w.r.t. the transitive K and let $b, c \in K$. Then $b \subset K$, and so $[\exists x \in b\Phi(x, b, c)]^K$ iff $\exists x \in b \cap K\Phi^K(x, b, c)$ iff $\exists x \in b\Phi(x, b, c)$. I.e.: the formula $\exists x \in y\Phi(x, y, z)$ is absolute as well. □

Lemma 7.12 *There are bounded formulas expressing the following:*

1. $x = \emptyset$,
2. $x \subset y$,
3. $z = \{x\}$, $z = \{x, y\}$,
4. $z = x \cup y$, $z = x \cup \{y\}$,
5. $x = 0$, $x = 1$, $x = 2$, $x = 3, \ldots$,
6. $x = \mathrm{V}_0$, $x = \mathrm{V}_1$, $x = \mathrm{V}_2$, $x = \mathrm{V}_3, \ldots$,
7. x *is* $0, \mathrm{S}$-*closed (i.e.,* $\emptyset \in x \wedge \forall y \in x(y \cup \{y\} \in x)$*),*
8. $x \in \mathrm{OR}$ *(cf. Exercise 130),*
9. α *is a limit ordinal,* α *is a successor ordinal,*

10. $x = \omega$, $x \in \omega$,

11. $y = \bigcup x$,

12. $z = (x, y)$ $(= \{\{x\}, \{x, y\}\})$,

13. p *is an ordered pair,*

14. R *is a relation,* xRy,

15. f *is a function,* $f(x) = y$, f *is an injection, resp., surjection, resp., bijection,*

16. $x = \mathrm{Dom}(f)$, $y = \mathrm{Ran}(f)$, $g = f|A$.

The statement of Lemma 7.12 is slightly vague. For instance, it is not always true that a straightforward elimination of the defined notions results in a formula with an obvious bounded logical equivalent. For many items (examples: 2, 3, 4, 10) this is true, but 9: "$x \in \omega$" has been defined as "x is in every $(0, \mathrm{S})$-closed set", which is certainly not bounded but Π_1 (see below). Nevertheless, applications always consist of assuming that items in the above list are given by a bounded formula (and, hence, are absolute).

Exercises

134 ♣ Prove (a few items of) Lemma 7.12.

135 ♣ Assume that K is transitive and that $\omega \in K$. Show that the Infinity Axiom ("there exists a $(0, \mathrm{S})$-closed set") holds in K.

136 ♣

1. Decide which ZF Axioms/Axiom schemas hold in V_ω, and which are false.

2. Same question for $\mathrm{V}_{\omega+\omega}$.

3. Same question for V_{ω_1}.

4. Obtain some relative consistency results from 1–3.

5. What about the truth of Theorem 4.10 (p. 27) (*every well-ordering has a type*) in the above models?

6. Suppose that Theorem 4.10 holds in V_α and $\alpha > \omega$. Can you give lower bounds for α? And if AC holds in V_α?

137 ♣ If you didn't solve Exercise 120 (p. 46), do it now. (Show: if $\kappa + 10 \leqslant \gamma$, and ($\kappa$ is strongly inaccessible)$^{\mathrm{V}_\gamma}$, then κ is, in fact, strongly inaccessible. Can you replace 10 by something smaller?)

138 ♣ Do *not* assume AC. Suppose that in V_α all ZF axioms hold. Show that α is an uncountable initial number. Show that $\alpha = \omega_\alpha$.

Definition 7.13 Formulas of the form $\forall x_1 \cdots \forall x_n \Phi$ where Φ is bounded are called Π_1; similarly, formulas of the form $\exists x_1 \cdots \exists x_n \Phi$ where Φ is bounded are called Σ_1.

Σ_1^{ZF} (resp., Π_1^{ZF}) is the collection of all set-theoretic formulas that, relative to ZF, possess a Σ_1 (resp., a Π_1) equivalent. (Analogously, in Σ_1^{ZFC} are the formulas that, relative to ZFC, possess a Σ_1 equivalent, etc.)

$\Delta_1^{\mathrm{ZF}} = \Pi_1^{\mathrm{ZF}} \cap \Sigma_1^{\mathrm{ZF}}$. (N.B.: $\Pi_1 \cap \Sigma_1 = \Delta_0$!)

Lemma 7.14 *Suppose that K is transitive, $\Phi(x_1, \ldots, x_n)$ is Π_1, $\Psi(x_1, \ldots, x_n)$ is Σ_1, and $a_1, \ldots, a_n \in K$. Then the following are true:*

1. $\Phi(a_1, \ldots, a_n) \rightarrow \Phi^K(a_1, \ldots, a_n)$ *(Π_1-formulas persist downward),*
2. $\Psi^K(a_1, \ldots, a_n) \rightarrow \Psi(a_1, \ldots, a_n)$ *(Σ_1-formulas persist upward),*

Proof. Trivial using Lemma 7.11. □

Corollary 7.15 *Δ_1^{ZF}-formulas are absolute w.r.t. transitive models of* ZF.

139 ♣ Show that the following conditions are Δ_1^{ZF}.

1. A is finite,
2. R is a well-ordering,
3. $a \in \mathrm{V}_\omega$, $X = \mathrm{V}_\omega$,
4. $y \in \mathrm{TC}(x)$, $y = \mathrm{TC}(x)$.

140 ♣ Show that the following are Π_1^{ZF} or Π_1^{ZFC}.

1. $y = \wp(x)$,
2. $x <_1 y$,
3. α is an initial number,
4. $\gamma < \mathrm{cf}(\alpha)$,
5. α is regular.

What about the following?:

1. $x \leqslant_1 y$,
2. $x =_1 y$,
3. $\alpha = \omega_1$,
4. $\beta = \mathrm{cf}(\alpha)$,
5. α is weakly inaccessible.

7.4 Constructible Hierarchy

The cumulative hierarchy involves *two* ingredients: the sequence of all ordinals, and the powerset operation. The constructible hierarchy modifies the cumulative hierarchy in that powersets are replaced by its definable version: whereas, in the cumulative hierarchy, a successor stage consists of *all* subsets of the previous stage, in the constructible hierarchy a successor stage only consists of the subsets that are *definable*.

The notion of definability used is the one from your logic course that employs the first-order satisfaction relation $\models$.

Definition 7.16 For a set A, $\mathrm{Def}(A)$ is the set of all $B \subset A$ such that for some set-theoretic formula $\varphi(x, y_1, \ldots, y_n)$ and elements $a_1, \ldots, a_n \in A$ we have that

$$B = \{a \in A \mid (A, \in) \models \varphi[a, a_1, \ldots, a_n]\}\,.$$

A completely precise version of this definition is 7.50 (on p. 71). Note that this definition must be thought of as *given in the context of ZF*.

Do you feel uneasy by this use of $\models$ from logic in the present set-theoretic context? Note that, in the above definition of Def, there occurs an existential quantifier over formulas. Thus, formulas must be thought of as sets here (since only sets can be values of variables). Therefore, the formulas of this definition must be distinguished from the formulas that we have discussed up to now: these were genuine, concrete things that may be written down. So maybe we should call formulas* the objects referred to in Definition 7.16. Hopefully, the explanations you'll get in Section 7.6 (starting from Definition 7.41 p. 68) will bring you the necessary comfort.

Satisfaction and formulas* will be defined eventually in such a way that, to every genuine formula $\Phi(x, x_1, \ldots, x_n)$, (it is provable in ZF that) there exists a formula* $\varphi(x, x_1, \ldots, x_n)$ (mimicking Φ) such that for all sets A and $a, a_1, \ldots, a_n \in A$ we have that

$$(A, \in) \models \varphi[a, a_1, \ldots, a_n] \;\text{ iff }\; \Phi^A(a, a_1, \ldots, a_n).$$

This is known as *Tarski's adequacy criterion* (see Lemma 7.47 p. 70) which expresses that $\models$ has been defined in the way intended.

As a consequence, we have the

Corollary 7.17 *1.* $\mathrm{Def}(A) \subset \wp(A)$*,*

2. for every genuine formula $\Phi(x, x_1, \ldots, x_n)$*, we have that*

$$a_1, \ldots, a_n \in A \;\Rightarrow\; \left\{a \in A \mid \Phi^A(a, a_1, \ldots, a_n)\right\} \in \mathrm{Def}(A).$$

Proof. This needs the precise definition of Def in Definition 7.50.2 p. 71. See Exercise 172 p. 71. □

Note: In this section and the next one, the only properties of Def that will ever be used are those of Corollary 7.17. They suffice for showing that all ZF axioms hold in L. However, verifying the Constructibility Axiom ($\mathbf{V} = \mathbf{L}$) in $\mathbf{L}$ needs a precise analysis of the definition of (Def and) $\mathbf{L}$.

Corollary 7.18 *1.* $\emptyset, A \in \mathrm{Def}(A)$*,*

2. *if* $a_1, a_2, a_3 \in A$, *then* $\{a_1, a_2, a_3\} \in \mathrm{Def}(A)$
   (and similarly with $1, 2, 4, 5, \ldots$ *replacing 3).*

Note: Using the above assumptions on Def only, it is impossible to prove the single statement "every finite subset of A is in $\mathrm{Def}(A)$" (Cf. Exercise 146).

Definition 7.19 The following recursion defines the *constructible hierarchy*:

1. $\mathrm{L}_0 = \emptyset$,
2. $\mathrm{L}_{\alpha+1} = \mathrm{Def}(\mathrm{L}_\alpha)$,
3. $\mathrm{L}_\gamma = \bigcup_{\xi<\gamma} \mathrm{L}_\xi$ (for limits γ).

N.B.: if you do not like to apply Def to $\emptyset$, restrict recursion equation 2 to $\alpha \geqslant 1$ and add the equation $\mathrm{L}_1 = \{\emptyset\}$.

The collection $\mathbf{L} = \bigcup_{\alpha\in\mathrm{OR}} \mathrm{L}_\alpha$ is the *constructible universe*; its elements are the *constructible* sets.

Compare the cumulative hierarchy that (assuming Foundation) generates *all* sets. Its main ingredients are the operation $\wp$ and the sequence OR. Of these two, the "wild" operation $\wp$ has been replaced here by the "tame" one Def of (first-order) logic. Only the presence of OR accounts for the fact that $\mathbf{L}$ is not a purely logical concept. Nevertheless, the replacement of $\wp$ by Def, according to most set theorists, makes truth of the constructibility axiom $\mathbf{V} = \mathbf{L}$ rather doubtful. (Why would every set eventually be captured by the rather accidental and arbitrary notion of first-order definability?) (On the other hand, "wildness" of OR could make up for the loss of $\wp$. However, strong infinity axioms that imply a lengthy OR never imply $\mathbf{V} = \mathbf{L}$ but often yield $\mathbf{V} \neq \mathbf{L}$.)

Exercises

A couple of easy properties of the constructible hierarchy are given by the following exercises.

141 ♣ $\mathrm{L}_\alpha \subset \mathrm{V}_\alpha$.

142 ♣ Every L_α is transitive. $\mathbf{L}$ is transitive.

143 ♣ If $\alpha < \beta$, then $\mathrm{L}_\alpha \in \mathrm{L}_\beta$, and, hence, $\mathrm{L}_\alpha \subset \mathrm{L}_\beta$.

144 ♣ $\mathrm{OR} \cap \mathrm{L}_\alpha = \alpha \in \mathrm{L}_{\alpha+1} - \mathrm{L}_\alpha$; $\mathrm{OR} \subset \mathbf{L}$.

Thus, all ordinals are constructible.

You'll see later that every finite subset of A is in $\mathrm{Def}(A)$. (Note that Corollary 7.18.2 only states this for finite subsets of concrete magnitude.) The claim for "arbitrary" finite subsets *cannot* be proved on the basis of 7.17 or 7.18, but needs a precise definition of Def. (See Exercise 174.) As a consequence, if $\alpha \leqslant \omega$, then $\mathrm{L}_\alpha = \mathrm{V}_\alpha$. However, you shall see later that $\mathrm{L}_{\omega+1} \neq \mathrm{V}_{\omega+1}$. (Why do you think this is so?)

What can be shown at this point is the following:

145 ♣

1. If $a \subset \mathrm{L}_\omega$ is finite, then $a \in \mathrm{L}_\omega$.

2. Hence, $\mathrm{V}_\omega \subset \mathrm{L}_\omega$; hence, $\mathrm{L}_\omega = \mathrm{V}_\omega$.

146 ♣ ♣ An "ω-incompleteness phenomenon".

1. The properties of Def from Corollary 7.17 suffice to show, for every *specific* natural number n, that $\mathrm{L}_n = \mathrm{V}_n$.

2. Show: if ZF is consistent, then it stays consistent upon the addition of (i) the properties of Def from Corollary 7.17 (*not* the definition of Def !), and (ii) the statement $\exists n \in \omega(\mathrm{L}_n \neq \mathrm{V}_n)$.

Hint. For 2: Note that 7.17 consists of infinitely many statements, one for each ZF-formula $\Phi(x, x_1, \ldots, x_n)$; and if A has m elements, then there can be at most m^n sets of the form $\{a \in A \mid \Phi^A(a, a_1, \ldots, a_n)\}$. Use the Compactness Theorem.

147 ♣ Foundation, Extensionality and Infinity Axioms hold in $\mathbf{L}$.
Hint. Exercises 128, 129, 135.

148 ♣ If $a \in \mathrm{L}_\alpha$, then $\bigcup a \in \mathrm{L}_{\alpha+1}$.
Thus $\mathbf{L}$ is closed under sumsets; *thus* the Sumset Axiom holds in $\mathbf{L}$.
Hint. Exercise 132.

149 ♣ If $a \subset \mathbf{L}$ is a set, then for some $\alpha \in \mathrm{OR}$, $a \subset \mathrm{L}_\alpha$.
Hint. Define the operation $h : a \to \mathrm{OR}$ by $h(x) =$ the least α s.t. $x \in \mathrm{L}_\alpha$; now consider (Substitution and Sumset Axiom) $\alpha = \bigcup\{h(x) \mid x \in a\}$.

150 ♣

1. If $\mathbf{L} \cap \wp(a) \subset \mathrm{L}_\alpha$, then $\mathbf{L} \cap \wp(a) \in \mathrm{L}_{\alpha+1}$.

2. If $a \in \mathbf{L}$, then $\mathbf{L} \cap \wp(a) \in \mathbf{L}$.

3. *Thus*, the Powerset Axiom holds in $\mathbf{L}$. (See Exercise 133.)

By part 2 of this Exercise, if $a \in \mathrm{L}_\alpha$, then for some β, $\mathbf{L} \cap \wp(a) \in \mathrm{L}_\beta$; but the proof (that involves Exercise 149) provides no information on the size of β. The GCH proof consists of supplying an upper bound for such a β: see Theorem 7.55 (p. 74).

For $\mathbf{L}$ to model all ZF axioms, it remains to see that the Substitution Axiom is true in it. To verify this, this axiom schema is "split" into (i) Separation Axiom and (ii) *Collection Principle*, which is the schema $\forall x \in a \exists y \Phi \to \exists b \forall x \in a \exists y \in b \Phi$ (b not free in Φ, Φ allowed to contain parameters next to the free variables x and y).

151 ♣ Show that Collection holds in $\mathbf{L}$.
Hint. Compare the hint given for Exercise 149.

152 ♣ Prove that Collection is a theorem (schema) in ZF.

153 ♣ Derive Substitution from Separation and Collection.

Thus, the only ZF axiom that remains to be checked in **L** is Separation. For this, we use the *Reflection Principle* (Theorem 7.20).

Here is the proof. Suppose that $a, b, c \in \mathbf{L}$ and that $\Phi = \Phi(x, y, z)$ is a formula. Separation in **L** requires that $d \in \mathbf{L}$ exists such that $[\forall x(x \in d \leftrightarrow x \in a \wedge \Phi(x, b, c))]^{\mathbf{L}}$. Clearly, there is at most one such d that satisfies this requirement: $d = \{x \in a \mid \Phi^{\mathbf{L}}(x, b, c)\}$. So, the problem boils down to showing that this set is in **L**.

Choose α big enough in order that $a, b, c \in \mathrm{L}_\alpha$. Then by definition of the constructible hierarchy, $d' = \{x \in \mathrm{L}_\alpha \mid [x \in a \wedge \Phi(x, b, c)]^{\mathrm{L}_\alpha}\}$ is an element of $\mathrm{L}_{\alpha+1}$. Obviously, $d' = \{x \in a \mid \Phi^{\mathrm{L}_\alpha}(x, b, c)\}$. Thus, it suffices to have α satisfy $\forall x, y, z \in \mathrm{L}_\alpha \left[\Phi^{\mathrm{L}_\alpha}(x, y, z) \leftrightarrow \Phi^{\mathrm{L}}(x, y, z)\right]$.

The existence of arbitrarily large α satisfying this (for arbitrary formulas Φ) is asserted by the *Reflection Principle* 7.20.

Pending the proof of this (and the precise explanation of what is meant by the operation Def together with Tarski's Criterion), what has been shown in this section is part of Theorem 7.1 (p. 52): **L** is a transitive inner model of ZF in ZF that contains all ordinals.

7.5 Reflection and Löwenheim-Skolem

Theorem 7.20 (Reflection for the constructible hierarchy)
For every formula $\Phi = \Phi(x_1, \dots, x_k)$, there are arbitrarily large ordinals α such that

$$\forall a_1, \dots, a_k \in \mathrm{L}_\alpha \left(\Phi^{\mathrm{L}_\alpha}(a_1, \dots, a_k) \leftrightarrow \Phi^{\mathbf{L}}(a_1, \dots, a_k)\right).$$

I.e.: for every ordinal β there is an ordinal $\alpha \geq \beta$ such that Φ is absolute between L_α and **L** (see Definition 7.8 p. 55).

The proof of Theorem 7.20 uses only two properties of the constructible hierarchy: (i) every constructible set is in some constructible level, and (ii) a union of a set of constructible levels is again a constructible level.

Similarly:

Theorem 7.21 (Reflection for the cumulative hierarchy)
For every formula $\Phi = \Phi(x_1, \dots, x_k)$, there are arbitrarily large ordinals α such that

$$\forall a_1, \dots, a_k \in \mathrm{V}_\alpha \left(\Phi^{\mathrm{V}_\alpha}(a_1, \dots, a_k) \leftrightarrow \Phi(a_1, \dots, a_k)\right).$$

I.e.: there are arbitrarily large ordinals α such that Φ is absolute w.r.t. V_α (absolute between V_α and **V**).

The proof of Theorem 7.21 uses only two properties of the cumulative hierarchy: (i) (Foundation) every set is in some partial universe, and (ii) a union of a set of partial universes is again a partial universe.

To prove both theorems simultaneously, it is worthwile to consider, instead of a *single* formula Φ, a finite bunch of ZF-formulas Σ that is *subformula-closed* (that is: every subformula of a formula in Σ is also in Σ). In particular, if Σ is the (finite!) collection of all subformulas of the initial formula Φ (together with Φ itself), then Theorems 7.20 and 7.21 immediately follow from Lemmas 7.24 and 7.23 below.

For the sequel of this section, Σ is a fixed, finite subformula-closed collection of formulas.

Let W be any class. A subclass $A \subset W$ is called a Σ-*elementary subcollection of* W, notation: $A \prec_\Sigma W$, if all sentences

$$\forall a_1, \ldots, a_k \in A \left(\Phi^A(a_1, \ldots, a_k) \leftrightarrow \Phi^W(a_1, \ldots, a_k)\right)$$

where $\Phi = \Phi(x_1, \ldots, x_k) \in \Sigma$, are satisfied; that is: if every formula in Σ is absolute between A and W.

Instead of Theorems 7.20 and 7.21, we show below that there are arbitrarily large α such that $\mathrm{L}_\alpha \prec_\Sigma \mathbf{L}$, resp., $\mathrm{V}_\alpha \prec_\Sigma \mathbf{V}$.

The following lemma will be helpful.

Lemma 7.22 *Suppose that $A \subset W$. In order for $A \prec_\Sigma W$ to hold, it suffices that the following (Tarski) condition is satisfied:*
for every existential quantification $\exists x \Phi(x, x_1, \ldots, x_k)$ in Σ and all $a_1, \ldots, a_k \in A$,
if $(\exists x \Phi)^W(x, a_1, \ldots, a_k)$ holds,
then $a \in A$ exists such that $\Phi^W(a, a_1, \ldots, a_k)$ holds as well.

Proof. Completely straightforward. We show, for $\Phi = \Phi(x_1, \ldots, x_k) \in \Sigma$, that

$$\forall a_1, \ldots, a_k \in A \left(\Phi^A(a_1, \ldots, a_k) \leftrightarrow \Phi^W(a_1, \ldots, a_k)\right)$$

using induction w.r.t. Φ. (The reason for taking Σ subformula-closed is to allow this induction; the condition of the lemma turns out to be the "missing link" in the induction.) W.l.o.g. we may assume that formulas in Σ are built using the logical operations $\neg, \wedge, \exists$ only. If $\Phi \in \Sigma$ is atomic, the equivalence is trivial. The induction steps for $\neg$ and $\wedge$ present no problem. Finally, if $\Phi = \exists x \Psi(x, y, z) \in \Sigma$ and $b, c \in A$ then: ($\Rightarrow$) If $\exists x \in A \Psi^A(x, b, c)$, say, $a \in A$ is s.t. $\Psi^A(a, b, c)$, then, by IH on Ψ (since $\Psi \in \Sigma$): $\Psi^W(a, b, c)$; thus, $\exists x \in W \Psi^W(x, b, c)$. Conversely ($\Leftarrow$): if $\exists x \in W \Psi^W(x, b, c)$, then, by the Tarski condition, $a \in A$ exists s.t. $\Psi^W(a, b, c)$; hence, by IH on Ψ, $\Psi^A(a, b, c)$ and $\exists x \in A \Psi^A(x, b, c)$. $\square$

The following implies Reflection for the cumulative hierarchy.

Lemma 7.23 *For every set B, there is a partial universe $\mathrm{V}_\alpha \supset B$ such that $\mathrm{V}_\alpha \prec_\Sigma \mathbf{V}$.*

Proof. Define a sequence of partial universes $A_0 \subset A_1 \subset A_2 \subset \cdots$ such that
(i) $A_0 \supset B$,
and, assuming the sequence $A_0 \subset \cdots \subset A_n$ defined, the partial uiniverse $A_{n+1} \supset A_n$ is chosen so large as to satisfy
(ii) for every existential quantification $\exists x \Phi(x, x_1, \ldots, x_k)$ in Σ and $a_1, \ldots, a_k \in A_n$: if $\exists x \Phi(x, a_1, \ldots, a_k)$ is true, then for some $a \in A_{n+1}$, $\Phi(a, a_1, \ldots, a_k)$ is true as well.

To be specific, A_{n+1} is constructed as follows. Fix an existential $\Psi = \exists x \Phi(x, x_1, \ldots, x_k)$ in Σ. For every finite sequence $s = (a_1, \ldots, a_k)$ from A_n for which $\exists x \Phi(x, a_1, \ldots, a_k)$ happens to be true, let V_s be the least partial universe that contains an example-x, i.e., for which $\exists x \in \mathrm{V}_s \Phi(x, a_1, \ldots, a_k)$ is true. Let V_Ψ be the union of the V_s's. (This uses both Sumset and Substitution Axiom.) Now, A_{n+1} is the union of A_n and the V_Ψ, where Ψ is an existential quantification in Σ.

It now follows from Lemma 7.22 that (the partial universe!) $A = \bigcup_n A_n$ satisfies the required conditions. $\square$

Obviously, we can similarly prove the following lemma, which implies Reflection for the constructible hierarchy.

Lemma 7.24 *For every set $B \subset \mathbf{L}$, there is a constructible level* $\mathrm{L}_\alpha \supset B$ *such that* $\mathrm{L}_\alpha \prec_\Sigma \mathbf{L}$.

The following classic from model theory can be proved using the same method.

Theorem 7.25 (Downward Löwenheim-Skolem-Tarski Theorem) *Suppose that W is an infinite set, that $\wp(W)$ has a choice function, and that $B \subset W$. Then $A \subset W$ exists such that $B \subset A$, $|A| = |B| + \aleph_0$, and $A \prec_\Sigma W$.*

Proof. Again we put $A = \bigcup_n A_n$, where the sequence $B \subset A_0 \subset A_1 \subset A_2 \subset \cdots \subset W$ satisfies, next to the condition on existentially quantified formulas in Σ from the previous proofs, the cardinality condition $|A_n| = |B| + \aleph_0$. For this, the choice function (equivalently, a well-ordering) is needed. For instance, if B is countable, and we have a countably infinite A_n, then A_{n+1} can be obtained by adding at most countably infinite "witnesses" to A_n (picked by the choice function): one for each (of the finitely many) existential quantification from Σ and each (of the countably many) finite sequence of parameters from A_n that satisfies it. □

Remark. Theorem 7.25 can be strengthened to the more usual formulation where Σ is the (infinite) set of all set-theoretic formulas. (And again, this has a generalization for arbitrary first-order languages.) But then the precise wording of the theorem requires the apparatus developed in Section 7.6; in particular, the notion of satisfaction is needed. For the Reflection Theorems, such generalizations can only make sense if we can define satisfaction over the proper classes V or **L** (provably impossible in the case of V).

Corollary 7.26 (Montague) ZF *is not finitely axiomatizable, provided it is consistent.*

Proof. Suppose that Φ is such that $\Phi \vdash$ZF. We show that Φ must be contradictory.

Using Φ as an axiom, argue as follows. Since Φ holds, we have all ZF axioms true, and, hence, the Reflection Principle as well. So the following corollary to Reflection holds:

$$\Phi \rightarrow \exists a(a \text{ transitive } \wedge \Phi^a).$$

Since Φ is true, there are transitive sets a such that Φ holds in it. By Foundation, let b be a *minimal* transitive set in which Φ holds. Applying the above corollary in b, we obtain $(\exists a(a \text{ transitive } \wedge \Phi^a))^b$, i.e. ("transitive" is absolute and $a \cap b = a$): $\exists a \in b(a \text{ transitive } \wedge \Phi^a)$, contradicting the choice of b. □

Definition 7.27 A collection $C \subset \mathrm{OR}$ is called

1. *unbounded* if it contains arbitrarily large ordinals; that is: if $\forall\alpha\exists\beta \in C(\alpha \leqslant \beta)$,
2. *unbounded in* a limit ordinal γ if it contains arbitrarily large ordinals $< \gamma$; that is: if $\forall\alpha < \gamma\exists\beta < \gamma(\beta \in C \wedge \alpha \leqslant \beta)$,
3. *closed* (in α) if it contains every limit ordinal ($< \alpha$) in which it is unbounded,
4. *closed unbounded* or *club* if it is both closed and unbounded.

Lemma 7.28 *1. The intersection of two clubs is a club,*

2. if each C_x (for every element x of a set a) is club, then so is $\bigcap_{x\in a} C_x$,

3. if each C_ξ ($\xi \in \mathrm{OR}$) is club, then so is $\left\{\alpha \in \mathrm{OR} \mid \alpha \in \bigcap_{\xi<\alpha} C_\xi\right\}$.

Exercises

154 ♣ Show that Reflection in the form of Lemma 7.23:

For every set B, there is a set $A \supset B$ such that $A \prec_\Sigma \mathbf{V}$.

(where Σ is any finite set of formulas) implies both Infinity Axiom and Collection Principle (relative to the other ZF axioms).

155 ♣ Show that ZF (provided consistent) is not finitely axiomatizable over *Zermelo set theory* Z (axiomatized by all axioms except Substitution). That is: there is no sentence Φ consistent with Z such that Z+Φ proves (all instances of) the Substitution Axiom.
Hint. All axioms of Z are true in every V_γ where $\gamma > \omega$ is a limit.

156 ♣ Show that the following are club: OR, the class of limit ordinals, the class of initials, of limit initials, the class $\{\alpha \mid \alpha = \omega_\alpha\}$.

157 ♣ Prove Lemma 7.28.

158 ♣ Suppose that $\omega < \mathrm{cf}(\alpha)$, $\beta < \mathrm{cf}(\alpha)$ and, for $\xi < \beta$, that C_ξ club in α (i.e.: closed in α, containing every limit $\gamma < \alpha$ of its elements). Show that $\bigcap_{\xi<\beta} C_\xi$ is club in α.

159 ♣ Let Σ be a finite subformula-closed set of formulas. Show that $\{\alpha \mid \mathrm{V}_\alpha \prec_\Sigma \mathbf{V}\}$ and $\{\alpha \mid \mathrm{L}_\alpha \prec_\Sigma \mathbf{L}\}$ are closed.

A precise formulation of the following exercises need the apparatus of Section 7.6 and the satisfaction definition 7.46. Also, see Remark 7.48 (p. 71).
N.B.: $A \prec B$ means $A \prec_\Sigma B$, where Σ is the set of *all* formulas.

160 ♣ Suppose that the initial λ is strongly inaccessible (Definition 6.24 p. 46). Show that $\alpha < \lambda$ exists such that $\mathrm{V}_\alpha \prec \mathrm{V}_\lambda$. Show that the smallest such α has $\mathrm{cf}(\alpha) = \omega$.

161 ♣ Assume that $\alpha < \beta$ and $\mathrm{V}_\alpha \prec \mathrm{V}_\beta$. Show that $\mathrm{V}_\alpha \models$ ZF.
Hint. Show that α is a limit $> \omega$. (This settles all axioms except Substitution.)

162 ♣ Suppose that an ordinal α exists such that $\mathrm{V}_\alpha \models$ ZF. Show that the smallest such α has $\mathrm{cf}(\alpha) = \omega$. (This *seems* to contradict the Substitution Axiom in V_α which implies that every operation : $\omega \to \alpha$ must be bounded. Thus, the function that witnesses countable cofinality of α cannot be definable in V_α.)

7.6 Consistency of $\mathbf{V} = \mathbf{L}$

Part of Theorem 7.1 has been proved now: in $\mathbf{L}$, all ZF axioms hold. This section shows that also $\mathbf{V} = \mathbf{L}$, the Axiom of Constructibility, holds in $\mathbf{L}$.

This may look trivial, but closer inspection shows it is not. That $\mathbf{V} = \mathbf{L}$ means, that $\forall x(x \in \mathbf{L})$. Thus, $[\mathbf{V} = \mathbf{L}]^{\mathbf{L}}$ means, that $\forall x \in \mathbf{L}[x \in \mathbf{L}]^{\mathbf{L}}$. Since $\forall x \in \mathbf{L}[x \in \mathbf{L}]$ holds trivially, it suffices to show that the formula $x \in \mathbf{L}$ is absolute w.r.t. $\mathbf{L}$ (i.e. —see Notation 7.9 p. 56—: that $\mathbf{L}^{\mathbf{L}} = \mathbf{L}$). This is one of the things shown here (see Exercise 163).

The heart of the matter is the following Theorem. See Definition 7.13 (p. 59) for Δ_1^{ZF}.

Theorem 7.29 *The formula $x = \mathrm{L}_\alpha$ is Δ_1^{ZF}.*

Proof. See, at the end of the present section, Exercise 173 (p. 71). □

Corollary 7.30 *The formula* $y \in \mathrm{L}_\alpha$ *is* Δ_1^{ZF}*, the formula* $y \in \mathbf{L}$ *is* Σ_1^{ZF}*.*

Proof. $y \in \mathrm{L}_\alpha \leftrightarrow \exists x(x = \mathrm{L}_\alpha \wedge y \in x)$;
$y \in \mathbf{L} \leftrightarrow \exists \alpha \in \mathrm{OR}(y \in \mathrm{L}_\alpha)$. □

Corollary 7.31 *(*ZF *proves that)* $\mathbf{V} = \mathbf{L}$ *holds in* $\mathbf{L}$*.*

Proof. We express $\mathbf{V} = \mathbf{L}$ as $\forall y \exists \alpha(y \in \mathrm{L}_\alpha)$. Then $[\mathbf{V} = \mathbf{L}]^{\mathbf{L}}$ is $\forall y \in \mathbf{L} \exists \alpha \in \mathbf{L}(y \in \mathrm{L}_\alpha)^{\mathbf{L}}$.
To prove this, let $y \in \mathbf{L}$ be arbitrary. Say, $y \in \mathrm{L}_\alpha$. Now $\alpha \in \mathbf{L}$ (for, $\mathrm{OR} \subset \mathbf{L}$). Applying Corollaries 7.15 (p. 59) and 7.30, we see that $(y \in \mathrm{L}_\alpha)^{\mathbf{L}}$ holds as well. □

From this, Corollary 7.2 (p. 52) (*if* ZF *is consistent, then so is* ZF+$\mathbf{V} = \mathbf{L}$) follows.

163 ♣ Exercise Show that $\mathbf{L}$ is absolute w.r.t. every transitive collection that contains all ordinals and satisfies sufficiently many ZF axioms.

Later on, we'll need the following *Condensation Lemma*:

Corollary 7.32 *Suppose that* $A \neq \emptyset$ *is transitive, that* $\alpha = \mathrm{OR} \cap A$ *is a limit and that* $[\mathbf{V} = \mathbf{L}]^A$ *holds. Then* $A = \mathrm{L}_\alpha$*.*

164 ♣ Prove Corollary 7.32.
Hint. Take the Axiom of Constructibility in the form $\forall y \exists \beta \exists x[y \in x \wedge x = \mathrm{L}_\beta]$, with an equivalent for "$x = \mathrm{L}_\beta$" of a suitable form.

Compare the cumulative hierarchy: from the Löwenheim-Skolem Theorem it follows that no sentence Φ exists that has arbitrarily large models of the form V_α such that from $A \neq \emptyset$, A transitive and Φ^A follows that $A = \mathrm{V}_\alpha$ for some α. (Cf. Exercise 193 p. 75.)

165 ♣ Exercise If K is a transitive collection such that $\mathrm{OR} \subset K$ and $[\mathbf{V} = \mathbf{L}]^K$, then $K = \mathbf{L}$.

The method by which Theorem 7.29 is proved looks as follows: scan all definitions leading up to the one of the constructible hierarchy and check that all notions defined are Δ_1^{ZF}-definable (and, hence, absolute).
This results in the following list. (The precise definition of the main notion Def also occurs in this list, cf. Definition 7.50.)
To smoothen things, we start with some closure properties for Σ_1^{ZF} and Δ_1^{ZF}.

Lemma 7.33 *Every* Σ_1^{ZF}*-definable operation is also* Δ_1^{ZF}*.*

Proof. If F is an operation, then $F(x) = y$ amounts to $\forall z(F(x) = z \to z = y)$. □

Lemma 7.34 *Any composition of* Σ_1^{ZF} *operations is* Σ_1^{ZF}*.*

Proof. For instance, $F(G(x)) = y$ amounts to $\exists z\,(G(x) = z \wedge F(z) = y)$. □

Lemma 7.35 *If* F *and* G *are* Σ_1^{ZF} *operations, then he expressions* $z \in F(x)$ *and* $G(u) \in F(x)$ *are* Δ_1^{ZF}*.*

Proof. $z \in F(x)$ is equivalent to both $\exists y(y = F(x) \wedge z \in y)$ and $\forall y(y = F(x) \rightarrow z \in y)$; $G(u) \in F(x)$ is equivalent to both $\exists z, y(z = G(u) \wedge y = F(x) \wedge z \in y)$ and $\forall z, y(z = G(u) \wedge y = F(x) \rightarrow z \in y)$. □

Lemma 7.36 *Σ_1^{ZF} is closed under bounded quantification.*

Proof. Immediate from the Collection Principle. □

Lemma 7.37 *If the operation F and the formula Φ are in Δ_1^{ZF}, then so are $\forall x \in F(y)\Phi$ and $\exists x \in F(y)\Phi$.*

Proof. $\forall x \in F(y)\Phi$ is equivalent with both $\exists z(z = F(y) \wedge \forall x \in z\, \Phi)$ and $\forall z(z = F(y) \rightarrow \forall x \in z\, \Phi)$. □

Lemma 7.38 *The class of Σ_1^{ZF}-operations is closed under $\in$-recursion.*

Proof. Suppose that $F(x) = H(F|x)$, where H is Σ_1^{ZF}. Then $F(x) = y$ holds iff a transitive set $a \ni x$ and a function f defined on a exist such that $\forall z \in a(f(z) = H(f|z))$, whereas $y = f(x)$. □

Lemma 7.39 *Suppose that F is defined by $F(a) = \{H(x) \mid x \in a\}$. If H is Σ_1^{ZF}, then so is F.*

Lemma 7.40 *The formulas $a \in \mathrm{V}_\omega$ and $X = \mathrm{V}_\omega$ are Δ_1^{ZF}.*

Proof. The first formula has a Σ_1-equivalent, since $a \in \mathrm{V}_\omega$ holds, iff:

$$\exists B(a \subset B \wedge B \text{ is transitive } \wedge B \text{ is finite }).$$

Thus, the second one is Δ_1^{ZF}, as $X = \mathrm{V}_\omega$ is true, iff:

$$\forall a \in X(a \in \mathrm{V}_\omega) \wedge \emptyset \in X \wedge \forall a, b \in X(a \cup \{b\} \in X).$$

Now a Π_1-equivalent for the first formula is:

$$\forall X\, [\emptyset \in X \wedge \forall u, v \in X(u \cup \{v\} \in X) \rightarrow a \in X]\,.$$

Check these equivalences! □

Now we start to inspect the series of definitions leading up to the one for Def and **L**.

The place to begin is to introduce formulas*: formulas as sets. The way to do this is more or less arbitrary. Often, formulas are seen as finite sequences of symbols built according to specific rules. However, things become simpler when formulatedin terms of the basic logical notions. We employ the following five simple (obviously: Σ_1^{ZF}) operations to simulate these logical operations.

Definition 7.41

$$\begin{aligned} v_i \doteq v_j &=_{\mathrm{def}} (0, i, j), \\ v_i \mathrel{\dot{\in}} v_j &=_{\mathrm{def}} (1, i, j), \\ \dot{\neg}\, \varphi &=_{\mathrm{def}} (2, \varphi), \\ \varphi \mathrel{\dot{\wedge}} \psi &=_{\mathrm{def}} (3, \varphi, \psi), \\ \dot{\exists} v_i\, \varphi &=_{\mathrm{def}} (4, i, \varphi). \end{aligned}$$

Note that in this definition, i, j, φ, ψ are variables. The curious notation suggests what we're aiming at.

With every genuine ZF formula Φ (let us, for simplicity's sake, assume that it only uses logical symbols $\neg$, $\wedge$ and $\exists$ and variables $x_0, x_1, x_2, \ldots$), we now can associate in a perfectly natural way a description $\ulcorner\Phi\urcorner$ of a formula* that "mimicks" Φ.

The first place where these descriptions are mentioned is in Lemma 7.47; the second and decisive one is where we prove Corollary 7.17 (p. 60) (see Exercise 172 p. 71).

To be completely precise, first associate descriptions $\ulcorner n\urcorner$ with natural numbers n, by

$\ulcorner 0\urcorner = \emptyset$,

$\ulcorner n+1\urcorner = \ulcorner n\urcorner \cup \{\ulcorner n\urcorner\}$.

(N.B.: Up to now, we never bothered to distinguish the "real world"-natural numbers 0, 1, 2,... from the sets $\emptyset$, $\{\emptyset\}$,... that play their roles!)

Next, define $\ulcorner\Phi\urcorner$ by means of the equations

$\ulcorner x_n = x_m\urcorner = v_{\ulcorner n\urcorner} \dot{=} v_{\ulcorner m\urcorner}$,

$\ulcorner x_n \in x_m\urcorner = v_{\ulcorner n\urcorner} \dot{\in} v_{\ulcorner m\urcorner}$,

$\ulcorner\neg\varphi\urcorner = \dot{\neg}\ulcorner\varphi\urcorner$,

$\ulcorner\varphi \wedge \psi\urcorner = \ulcorner\varphi\urcorner \dot{\wedge} \ulcorner\psi\urcorner$,

$\ulcorner\exists x_n\varphi\urcorner = \dot{\exists} v_{\ulcorner n\urcorner} \ulcorner\varphi\urcorner$.

That is: "replace x's by v's and put dots above the appropriate places".

Definition 7.42 1. A *formula** is something that belongs to every set X with the following two properties:

(a) $\forall i, j \in \omega [v_i \dot{=} v_j, v_i \dot{\in} v_j \in X]$,

(b) $\forall \varphi, \psi \in X \forall i \in \omega [\dot{\neg}\, \varphi, \varphi \dot{\wedge} \psi, \dot{\exists} v_i\, \varphi \in X]$.

2. FORM is the set of all formulas*.

Note that FORM $\subset \mathrm{V}_\omega$.

The usual principle of *formula induction* coincides with the principle of fixed point induction that goes with the definition of FORM as a least fixed point.

Lemma 7.43 *The formulas* $\varphi \in$ FORM *and* $X =$ FORM *are* Δ_1^{ZF}.

Proof. The first one obviously has a Π_1-equivalent. A Σ_1-equivalent says: there is a set A such that $\varphi \in A$, and such that every $b \in A$ either is an atomic formula* or is a negation, conjunction, or quantification of some formula(s)* in A.

A Σ_1-equivalent for the second one can be obtained via the equivalent $\forall a \in X(a \in \mathrm{FORM}) \wedge \forall a \in \mathrm{V}_\omega(a \in \mathrm{FORM} \to a \in X)$.

Alternatively, FORM is the union of the FORM_n, where FORM_n is the set of all formulas* built in n steps. The sequence of the FORM_n can be recursively defined. □

166 ♣ Define $A^{<\omega} = \{f \mid f \text{ is a finite function s.t. } \mathrm{Dom}(f) \subset \omega \wedge \mathrm{Ran}(f) \subset A\}$. Show that the formula $X = A^{<\omega}$ is Σ_1^{ZF}.

Lemma 7.44 *The function* FrV *from* FORM *to finite sets of natural numbers that produces the set of (indices of) free variables of a formula* is* Σ_1^{ZF}.

167 ♣ Prove this.

Definition 7.45 Define the operation SAT recursively as follows.
Note that, for all sets A and formulas φ, $\mathrm{SAT}(A,\varphi) \subset A^{\mathrm{FrV}(\varphi)}$.

1. $\mathrm{SAT}(A, v_i \doteq v_j) = \{f \in A^{\{i,j\}} \mid f(i) = f(j)\}$ $(= \{\{(i,a),(j,a)\} \mid a \in A\})$,
2. $\mathrm{SAT}(A, v_i \dot{\in} v_j) = \{f \in A^{\{i,j\}} \mid f(i) \in f(j)\}$,
3. $\mathrm{SAT}(A, \dot{\neg}\, \varphi) = A^{\mathrm{FrV}(\varphi)} - \mathrm{SAT}(A,\varphi)$,
4. $\mathrm{SAT}(A, \varphi \mathbin{\dot{\wedge}} \psi)$
   $= \{f \in A^{\mathrm{FrV}(\varphi)\cup\mathrm{FrV}(\psi)} \mid (f|\mathrm{FrV}(\varphi)) \in \mathrm{SAT}(A,\varphi) \ \wedge\ (f|\mathrm{FrV}(\psi)) \in \mathrm{SAT}(A,\psi)\}$,
5. $\mathrm{SAT}(A, \dot{\exists} v_i\, \varphi) = \begin{cases} \{f - \{(i, f(i))\} \mid f \in \mathrm{SAT}(A,\varphi)\} & \text{if } i \in \mathrm{FrV}(\varphi) \\ \mathrm{SAT}(A,\varphi) & \text{otherwise} \end{cases}$

The recursion here is along $\in^\star$ (the transitive closure of $\in$) which is well-founded (since we're assuming Foundation): note that, by Definition 7.41, $\varphi \in^\star \dot{\neg}\, \varphi$; $\varphi, \psi \in^\star \varphi \mathbin{\dot{\wedge}} \psi$; and $\varphi \in^\star \dot{\exists} v_i\, \varphi$.

Intuitively, $\mathrm{SAT}(A,\varphi)$ is the set of A-assignments for $\mathrm{FrV}(\varphi)$ that satisfy φ in $(A,\in)$. However, the satisfaction relation $\models$ is defined in terms of SAT, cf. Definition 7.46.

N.B.: Note that the Recursion Theorem 4.13 (p. 29) and the proof of Lemma 7.38 (p. 68) only apply here *when A is a set:* only then the values $\mathrm{SAT}(A,\varphi)$ are sets as well. If A is a proper class, nothing guarantees that a similar operation exists.

In fact, Theorem 7.49 says that it is impossible to define an operation $\mathrm{SAT}(\mathbf{V},\varphi)$ on the universe $\mathbf{V}$ that has the same recursive properties (unless ZF is inconsistent). (However, satisfaction in $\mathbf{L}$ happens to be definable under strong infinity assumptions that, of course, imply $\mathbf{V} \neq \mathbf{L}$ as well.)

168 ♣ Show that SAT is Σ_1^{ZF}.

Definition 7.46 $A \models \varphi[f]$ iff: $\varphi \in \mathrm{FORM}$, f is a function such that $\mathrm{FrV}(\varphi) \subset \mathrm{Dom}(f) \subset \omega$ and $\mathrm{Ran}(f) \subset A$, and $f|\mathrm{FrV}(\varphi) \in \mathrm{SAT}(A,\varphi)$.

169 ♣ Show that $\models$ is Δ_1^{ZF}.

Lemma 7.47 (Tarski's Correctness Condition) *For every genuine formula $\Phi(x_0,\dots,x_{n-1})$ (of the type considered)* ZF *proves:*

for every set A and every $f : \ulcorner n \urcorner \to A$ we have:

$$\Phi^A\left(f(\ulcorner 0 \urcorner), \dots, f(\ulcorner n-1 \urcorner)\right) \ \Leftrightarrow\ A \models \ulcorner \Phi \urcorner [f].$$

170 ♣ Prove this.
Hint. First, show that if $A \neq \emptyset$, $\varphi, \psi \in \mathrm{FORM}$, and $f \in A^{<\omega}$ is appropriate, then:

1. $A \models v_i \doteq v_j[f]$ iff $f(i) = f(j)$,
2. $A \models v_i \dot{\in} v_j[f]$ iff $f(i) \in f(j)$,
3. $A \models \dot{\neg}\, \varphi[f]$ iff $A \not\models \varphi[f]$,

4. $A \models (\varphi \mathbin{\dot\wedge} \psi)[f]$ iff both $A \models \varphi[f]$ and $A \models \psi[f]$,

5. $A \models \dot\exists v_i\, \varphi[f]$ iff $\exists a \in A(A \models \varphi[f^i_a])$
   Here, f^i_a is defined by $\mathrm{Dom}(f^i_a) = \mathrm{Dom}(f) \cup \{i\}$, $f^i_a(i) = a$, and, for $j \neq i$, $j \in \mathrm{Dom}(f)$: $f^i_a(j) = f(j)$.

Remark 7.48 Models vs. inner models. It is now straighforward to define a *set* zf $\subset$ FORM (Definition 7.42 p. 69) of formulas*, the *set of ZF-axioms*. A set A is a ZF-*model* by definition if it holds that $\forall\varphi \in \mathrm{zf}(A \models \varphi)$. Note that the latter property of A is expressed using *one* ZF-formula. In contrast, that a definable class K is an inner model of ZF (in ZF) means that *infinitely many* separate sentences (every sentence Φ^K where Φ is a ZF-axiom) are *provable in ZF*. If a set A is a ZF-model then (provided the definition of zf is faithful) by the Correctness Condition, every single ZF-axiom holds in A. But since there are infinitely many ZF-axioms, the converse of this doesn't make sense. However, note that the theory ZFC(**M**) of Subsection 8.1.1 (p. 77) remains consistent upon adding the sentence stating that **M** is *not* a ZF-model (Exercise 199).

Theorem 7.49 (Tarski; undefinability of satisfaction and truth.) *Unless* ZF *is inconsistent, there is no formula* $\Sigma(x, y)$ *such that for every formula* $\Phi(x_0, \ldots, x_{n-1})$, ZF *proves:*

for every every $f : \ulcorner n \urcorner \to \mathbf{V}$ *we have that:*

$$\Phi\left(f(\ulcorner 0 \urcorner), \ldots, f(\ulcorner n-1 \urcorner)\right) \;\Leftrightarrow\; \Sigma(\ulcorner \Phi \urcorner, f).$$

Proof. If such a Σ does exist, consider the formula $\Psi(x_0) \equiv \neg\Sigma(x_0, \{(\ulcorner 0 \urcorner, x_0)\})$ and look what happens with $\neg\Psi(\ulcorner \Psi \urcorner)$ and $f = \{(\ulcorner 0 \urcorner, \ulcorner \Psi \urcorner)\}$. (The argument compares well with the one of the Russell paradox.) □

Definition 7.50 1. $D(A, \varphi, f) = \{a \in A \mid A \models \varphi[f \cup \{(\ulcorner 0 \urcorner, a)\}]\}$
(where $f : (\mathrm{FrV}(\varphi) - \{\ulcorner 0 \urcorner\}) \to A\}$).

2. $\mathrm{Def}(A) = \{D(A, \varphi, f) \mid \varphi \in \mathrm{FORM} \wedge f : (\mathrm{FrV}(\varphi) - \{\ulcorner 0 \urcorner\}) \to A\}$.

The next three exercises accomplish what has been promised.

171 ♣ Show that these operations are Σ^{ZF}_1.

172 ♣ Prove Corollary 7.17 (p. 60): if $\Phi(x_0, x_1, \ldots, x_n)$ is a genuine formula and $a_1, \ldots, a_n \in A$, then $\{a_0 \in A \mid \Phi^A(a_0, a_1, \ldots, a_n)\} \in \mathrm{Def}(A)$.
Hint. Apply Lemma 7.47 to $\ulcorner \Phi \urcorner$ and all functions $\{(\ulcorner 0 \urcorner, a_0)\} \cup f$, where $f = \{(\ulcorner 1 \urcorner, a_1), \ldots, (\ulcorner n \urcorner, a_n)\}$.

173 ♣ Prove Theorem 7.29 (p. 66): the formula $x = \mathrm{L}_\alpha$ is in Δ^{ZF}_1.

174 ♣ Show: (if $A \neq \emptyset$, then) $\mathrm{Def}(A)$ contains all finite subsets of A.

175 ♣ Give the details of the proof for Theorem 7.49.

176 ♣ (Σ_1-definability of Σ_1-satisfaction. Compare Theorem 7.49.) Show: there is a Σ_1-formula $\Sigma(x, y)$ such that for every Σ_1-formula $\Phi(x_0, \ldots, x_{n-1})$ it is provable in ZF that

for every every $f : \ulcorner n \urcorner \to \mathbf{V}$ we have that:

$$\Phi(f(\ulcorner 0 \urcorner), \ldots, f(\ulcorner n-1 \urcorner)) \;\Leftrightarrow\; \Sigma(\ulcorner \Phi \urcorner, f).$$

Hint. $\Sigma(\varphi, f)$ is: $\exists A(A$ is transitive $\wedge\ \mathrm{Ran}(f) \subset A \ \wedge\ A \models \varphi[f])$.
What about (satisfaction of) Π_1-formulas?

7.7 $\mathbf{V} = \mathbf{L}$ implies AC

Here we prove the first half of Theorem 7.3: that, in ZF, $\mathbf{V} = \mathbf{L}$ implies AC. In fact, the Wellordering Theorem holds in a strong way: it is possible to define a wellordering (a linear, $\in$-like, well-founded relation) R on all of $\mathbf{L}$.

This wellordering R will be defined by

$$a\mathrm{R}b \equiv_{\mathrm{def}} \exists\alpha[(a \in \mathrm{L}_\alpha \wedge b \notin \mathrm{L}_\alpha) \vee (a, b \in \mathrm{L}_{\alpha+1} - \mathrm{L}_\alpha \wedge a\mathrm{R}_{\alpha+1}b)]$$

where each $\mathrm{R}_{\alpha+1}$ (to be defined) well-orders $\mathrm{L}_{\alpha+1}$.

Exercises

177 ♣ Show that, under the above definition (assuming that $\mathrm{R}_{\alpha+1}$ well-orders $\mathrm{L}_{\alpha+1}$), R is a linear, $\in$-like, well-founded relation on $\mathbf{L}$.

The results of the following Exercises 178–181 suffice to define the R_α.

178 ♣ Suppose that $(A, <)$ is a wellordering and $f : A \to B$ a surjection. Define the relation $\prec$ on B by $x \prec y \equiv_{\mathrm{def}}$ the $<$-first element of $f^{-1}(x)$ is $<$-smaller than the $<$-first element of $f^{-1}(y)$. Then $\prec$ wellorders B.

179 ♣ V_ω and, hence, FORM and the set of all finite subsets of ω can be wellordered in type ω. (See Exercise 87 p. 34.)

180 ♣ Suppose that $(A, <)$ and $(B, <')$ are wellorderings. Define the relation $\prec$ on $A \times B$ by $(a, b) \prec (a', b') \equiv_{\mathrm{def}} a < a' \vee (a = a' \wedge b <' b')$. Then $\prec$ is a wellordering.

181 ♣ Suppose that (A, S) is a wellordering. Define the relation $\prec$ on $A^{<\omega}$ by: $f \prec g$ iff $f \neq g$, and either: $\mathrm{Dom}(f)$ precedes $\mathrm{Dom}(g)$ in the wellordering on the finite subsets of ω that exists according to Exercise 179, or: $\mathrm{Dom}(f) = \mathrm{Dom}(g)$, and we have that $f(n)Sg(n)$ for the least $n \in \mathrm{Dom}(f)$ such that $f(n) \neq g(n)$. Then $\prec$ is a wellordering.

Definition 7.51 1. $\mathrm{R}_0 = \emptyset$,

2. $\mathrm{R}_{\alpha+1}$ is the wellordering of $\mathrm{L}_{\alpha+1}$ that can be constructed from R_α (assuming this is a wellordering of L_α) using Exercises 178–181.

   More precisely: assuming that R_α well-orders L_α, it induces a wellordering of $\mathrm{L}_\alpha^{<\omega}$ (Exercise 181), therefore we can define a well-ordering for FORM $\times\, \mathrm{L}_\alpha^{<\omega}$ (Exercises 179 and 180); and finally this produces the wellordering $\mathrm{R}_{\alpha+1}$ of $\mathrm{L}_{\alpha+1}$, since $\mathrm{L}_{\alpha+1}$ is a surjective image of FORM $\times\, \mathrm{L}_\alpha^{<\omega}$ (Exercise 178).

3. When γ is a limit, define, for $a, b \in \mathrm{L}_\gamma$ (compare the above definition for R):

   $a\mathrm{R}_\gamma b \equiv_{\mathrm{def}} \exists\xi < \gamma\,[(a \in \mathrm{L}_\xi \wedge b \notin \mathrm{L}_\xi) \vee (a, b \in \mathrm{L}_{\xi+1} - \mathrm{L}_\xi \wedge a\mathrm{R}_{\xi+1}b)]$.

182 ♣ Check the details of the construction.

7.8 V = L implies GCH

As Corollary 7.56 shows, Theorem 7.55 is the crucial result in the derivation of GCH from $\mathbf{V} = \mathbf{L}$.

Assume $\mathbf{V} = \mathbf{L}$. Since $\wp(\omega)$ is uncountable and L_α is countable as long as α is countable (cf. Lemma 7.53) there is no $\alpha < \omega_1$ such that $\wp(\omega) \subset \mathrm{L}_\alpha$. Thus, the bounding result $\wp(\omega) \subset \mathrm{L}_{\omega_1}$, which is a special case of Theorem 7.55, is sharp.

In order to prove Theorem 7.55, we need three preliminary facts, some of which you may know already.

The second part of the following lemma goes under the name "Mostowski's Collapsing Lemma". The first part is Exercise 67 (p. 29).

Recall that models (A, R) and (B, S) are *isomorphic* —notation: $(A, R) \cong (B, S)$— if there exists an *isomorphism* between these structures, that is: a bijection $h : A \to B$ such that for all $a, a' \in A$: $aRa' \Leftrightarrow h(a)Sh(a')$.

Lemma 7.52 1. *Every wellfounded (set-)model of the Extensionality Axiom is isomorphic with a model of the form $(B, \in)$ where B is transitive.*

2. *In particular (since by Foundation, $\in$ is wellfounded), every (set-)model $(A, \in)$ of the Extensionality Axiom is isomorphic with a model of the form $(B, \in)$ where B is transitive.*

   Moreover, if h is the isomorphism and $X \subset A$ is any transitive part of A, then $h|X$ is the identity on X.

Proof. 1. Assume that (A, R) is a well-founded model of the Extensionality Axiom. Using recursion along R (Theorem 4.13 p. 29), define a function h on A by $h(x) = \{h(y) \mid yRx\}$. Put $B = \mathrm{Ran}(h)$. It follows that B is transitive and (since R is extensional) that $h : A \to B$ is an isomorphism.

2. The first claim follows immediately from 1. For the second one, the implication $x \in X \Rightarrow h(x) = x$ follows using $\in$-induction. □

Lemma 7.53 *If $\omega \leqslant \alpha$, then $\mathrm{L}_\alpha =_1 \alpha$.*

183 ♣ Exercise Show this. Show that a proof must exist that does not use AC. (Cf. Exercise 189.)

A last detail is the following Lemma, which states that satisfaction is preserved by isomorphism. A similar result applies to genuine formulas and sentences; this can be proved directly but is also immediate from 7.54 and Lemma 7.47.

Lemma 7.54 *If h is an isomorphism between (A, R) and (B, S) then, for $\varphi \in$ FORM and $f : \mathrm{FrV}(\varphi) \to A$,*

$$(A, R) \models \varphi[f] \ \Leftrightarrow\ (B, S) \models \varphi[h \circ f].$$

Proof. Straightforward induction w.r.t. φ. (Or, see any introductory logic text.) □

The following is the main result of this section: it says at which constructible level the constructible powerset of ω_α is located. Cf. the remarks immediately after Exercise 150 (p. 62). The proof of this uses almost everything derived so far.

Theorem 7.55 $\mathbf{V} = \mathbf{L} \;\Rightarrow\; \wp(\omega_\alpha) \subset \mathrm{L}_{\omega_{\alpha+1}}$. *(In fact,* $\mathbf{L} \cap \wp(\omega_\alpha) \subset \mathrm{L}_{\omega_{\alpha+1}}$.*)*

Corollary 7.56 $\mathbf{V} = \mathbf{L} \;\Rightarrow\;$ GCH.

Proof. Assume $\mathbf{V} = \mathbf{L}$. By Theorem 7.55, $|\wp(\omega_\alpha)| \leqslant |\mathrm{L}_{\omega_{\alpha+1}}|$. Thus (using Lemma 7.53): $2^{\aleph_\alpha} = |\wp(\omega_\alpha)| \leqslant |\mathrm{L}_{\omega_{\alpha+1}}| = |\omega_{\alpha+1}| = \aleph_{\alpha+1}$.

The inequality $\aleph_{\alpha+1} \leqslant 2^{\aleph_\alpha}$ is a consequence of AC. □

Proof of Theorem 7.55. We only present the special case that (if $\mathbf{V} = \mathbf{L}$, then) $\wp(\omega) \subset \mathrm{L}_{\omega_1}$, which implies CH. (Exercise 184 asks you to prove the full result.) Two proofs are given. The second one may look slightly easier; it differs from the first only in not using the Condensation Lemma 7.32.

First proof.

Assume that $\mathbf{V} = \mathbf{L}$ and $a \in \wp(\omega)$. In order that $a \in \mathrm{L}_{\omega_1}$, we want an ordinal $\beta < \omega_1$ such that $a \in \mathrm{L}_\beta$.

Since $a \in \mathbf{L}$, an ordinal γ exists such that $a \in \mathrm{L}_\gamma$. (But γ may not satisfy $\gamma < \omega_1$.)

We have that $[\mathbf{V} = \mathbf{L} \wedge \forall\xi\exists\delta(\xi < \delta)]^{\mathbf{L}}$. By Reflection, choose $\lambda \geqslant \gamma$ such that $[\mathbf{V} = \mathbf{L} \wedge \forall\xi\exists\delta(\xi < \delta)]^{\mathrm{L}_\lambda}$.

According to the Downward Löwenheim-Skolem-Tarski Theorem 7.25, a countable set A exists such that $\{a\} \cup \omega \subset A \subset \mathrm{L}_\lambda$ and [Extensionality Axiom $\wedge\, \mathbf{V} = \mathbf{L} \wedge \forall\xi\exists\delta(\xi < \delta)]^A$ (we can get Extensionality in A since this holds in the transitive set L_λ).

Using the Mostowski Collapsing Lemma 7.52.2, let $h : A \to B$ be an $\in$-isomorphism onto a transitive set B. Since isomorphism preserves truth (Lemma 7.54), we also have that $[\mathbf{V} = \mathbf{L} \wedge \forall\xi\exists\delta(\xi < \delta)]^B$.

By the Condensation Lemma 7.32 (p. 67), $B = \mathrm{L}_\beta$, where $\beta = \mathrm{OR} \cap B$.

This is the ordinal we're looking for: $A =_1 B = \mathrm{L}_\beta =_1 \beta$ (Lemma 7.53), and, since $\{a\} \cup \omega$ is a transitive part of A, (according to 7.52.2) we have that $h(a) = a$ and so $a \in B = \mathrm{L}_\beta$. □

Second proof.

Again, assume that $a \in \wp(\omega)$. Again, we want an ordinal $\beta < \omega_1$ such that $a \in \mathrm{L}_\beta$.

Again, since $a \in \mathbf{L}$, an ordinal γ exists such that $a \in \mathrm{L}_\gamma$, where we may consider "$y \in \mathrm{L}_\gamma$" to be (given by) a Σ_1-formula.

Since $\mathbf{V} = \mathbf{L}$, we trivially have that $[a \in \mathrm{L}_\gamma]^{\mathbf{L}}$. By Reflection, choose $\lambda > \gamma$ such that $[a \in \mathrm{L}_\gamma]^{\mathrm{L}_\lambda}$. (This is an inessential use of Reflection: all that is needed is that L_λ contains witnesses for the existential quantifiers in the Σ_1-statement $a \in \mathrm{L}_\gamma$.)

According to the Downward Löwenheim-Skolem Theorem 7.25 applied to the model $(\mathrm{L}_\lambda, \in)$ and the countable subset $X = \omega \cup \{a, \gamma\}$, a countable set A exists such that $\omega \cup \{a, \gamma\} \subset A \subset \mathrm{L}_\lambda$, which satisfies as many statements (using the parameters a and γ) as you like — as long as they are true about L_λ.

In particular, we can assume Extensionality holds in A. Using the Mostowski Collapsing Lemma 7.52.2, let $h : A \to B$ be an $\in$-isomorphism onto a transitive set B.

Claim. $\beta =_{\mathrm{def}} h(\gamma)$ is the desired ordinal.

Proof. First, β is countable: $\beta \in B$, B is transitive, so $\beta \subset B$, and B is countable.

Next, $\beta \in \mathrm{OR}$. For, $\gamma \in \mathrm{OR}$. This fact is expressed by a bounded formula. Thus, it holds in L_λ as well. Thus it can be assumed to hold in A. Thus (since isomorphism preserves truth) $h(\gamma) \in \mathrm{OR}$ holds in B. By absoluteness, we have that $\beta = h(\gamma) \in \mathrm{OR}$.

Finally, truth of $a \in \mathrm{L}_\beta$ is obtained by a similar argument: λ was chosen so as to have $a \in \mathrm{L}_\gamma$ true in L_λ. Therefore, it can be assumed true in A. Thus, $h(a) \in \mathrm{L}_{h(\gamma)}$ is true in B. However, $h(\gamma) = \beta$; $\omega \cup \{a\}$ is a transitive part of A containing a and, therefore, $h(a) = a$. Thus, $a \in \mathrm{L}_\beta$ is true in B. By upward persistence of Σ_1-formulas, $a \in \mathrm{L}_\beta$ is true. □

Exercises

184 ♣ Prove Theorem 7.55 in full generality. Somewhat more general, show that, not assuming $\mathbf{V} = \mathbf{L}$: if $\omega \leqslant \alpha$, then $\mathbf{L} \cap \wp(\mathrm{L}_\alpha) \subset \mathrm{L}_{\Gamma(\alpha)}$. (N.B.: $\Gamma(\alpha)$ is the least initial $> \alpha$. Cf. Lemma 4.26 p. 35. In particular, $\Gamma(\omega_\beta) = \omega_{\beta+1}$.)

185 ♣ Note that the (Π_1) formula $y = \wp(x)$ is not logically equivalent with an Σ_1-formula. Proof: it is satisfied by $x = 2$ and $y = 3$ in $(\omega, \in)$, but not in $(\mathrm{V}_\omega, \in)$; whereas an Σ_1-formula would be upward preserved.

Show that the formula $y = \wp(x)$ is not Σ_1^{ZF} (provided ZF is consistent).

186 ♣ Show that the formula $x =_1 y$ (which is Σ_1^{ZF}) is not Π_1^{ZF} (unless ZF is inconsistent).

Show the same thing for the formula $x =_1 \omega$.

187 ♣ Show that L_{ω_1} models all ZF axioms, with the possible exception of the Powerset Axiom.

188 ♣ Show that $\{\alpha < \omega_1 \mid (\mathrm{L}_\alpha, \in) \prec (\mathrm{L}_{\omega_1}, \in)\}$ is club in ω_1.

189 ♣ Show:

1. Every Σ_1 statement provable in ZFC (or ZF+$\mathbf{V} = \mathbf{L}$) is also provable in ZF.
2. The same thing holds for statements of the form $\forall\alpha\in\mathrm{OR}\,\Phi(\alpha)$ where Φ is Σ_1.

190 ♣ (Levy) Assume that Φ is bounded. Show that ZFC proves:

$$\forall a\,[\exists b\Phi(a,b) \;\rightarrow\; \exists b\,(|\mathrm{TC}(b)| \leqslant |\mathrm{TC}(a)| + \aleph_0 \;\wedge\; \Phi(a,b))]\,.$$

Show this also for Φ that are Σ_1.

Hint. Needed: Löwenheim-Skolem, collapsing, persistence (you may like to use Reflection as well).

191 ♣ Give a fast proof of Theorem 7.55 using Exercise 190.

192 ♣ Assume $\mathbf{V} = \mathbf{L}$. Prove that $\{\alpha \in \mathrm{OR} \mid \mathrm{L}_\alpha = \mathrm{V}_\alpha\}$ is club.

193 ♣ There is no sentence Φ consistent with ZF for which the following holds: if $A \neq \emptyset$ is transitive, $\alpha = \mathrm{OR} \cap A$ is a limit, and Φ^A holds, then $A = \mathrm{V}_\alpha$. (Cf. the Condensation Lemma, Corollary 7.32 p. 67.)

194 ♣ Show: ZF+$\mathbf{V} = \mathbf{L} \vdash \Phi$ iff ZF$\vdash \Phi^{\mathbf{L}}$.

195 ♣ Assume that K is a transitive class in which all ZF axioms hold. Show that $\omega_1^K \leqslant \omega_1$.

Show that $\mathbf{L} \cap \wp(\omega) \subset \mathrm{L}_{\omega_1^{\mathbf{L}}}$.

196 ♣ Show that $\omega_1^{\mathbf{L}} < \omega_1$ iff $\mathbf{L} \cap \wp(\omega)$ is countable.

197 ♣

1. Assume that a set A exists such that $(A, \in)$ is a model of all ZF-axioms (considered as a certain subset of FORM).

   Show:

   (a) There is such a set A that is transitive.

   (b) There is such a set A that has the form L_α, where $\alpha < \omega_1$.

2. Assume that α is the least ordinal such that $(\mathrm{L}_\alpha, \in)$ is a ZF-model.

   Show that if A is a transitive set such that $(A, \in)$ is a ZF-model, then $\alpha \subset A$, and (hence) $\mathrm{L}_\alpha \subset A$.

   $(\mathrm{L}_\alpha, \in)$ is called the *minimal* (transitive) ZF-model.

It cannot be excluded that the minimal ZF-model exists. In that case, it is not possible to prove consistency of $\neg$AC, $\neg$GCH or $\mathbf{V} \neq \mathbf{L}$ (or, for that matter, any statement contradicting $\mathbf{V} = \mathbf{L}$) using an inner model. For, suppose that ZF $\vdash K \neq \emptyset \wedge K$ transitive $\wedge$ (ZF+$\mathbf{V} \neq \mathbf{L})^K$. Let M be the minimal model. Since $M \models$ZF, we get $M \models$ (ZF+$\mathbf{V} \neq \mathbf{L})^K$, i.e., $K^M \models$ZF+$\mathbf{V} \neq \mathbf{L}$. However, since M is minimal, $K^M = M$. But, $M \models \mathbf{V} = \mathbf{L}$. Contradiction.

The solution of this predicament consists in the forcing method, discovered by Cohen in 1963. There are several ways to view forcing: forcing not only modifies the universe (as does an inner model), but it also modifies its relation $\in$; alternatively: it preserves $\in$ but does not narrow down the universe, instead —surprisingly!— *extends* it.

To begin with, Cohen complemented Gödel's results by the following.

Theorem 7.57 *If* ZF *is consistent, then so are*

1. ZF + $\mathbf{V} \neq \mathbf{L}$

   (some $a \subset \omega$ may not be constructible, in fact, $\mathbf{L} \cap \wp(\omega)$ may be countable),

2. ZF + $\neg$AC

   (e.g., the reals may not have a wellordering),

3. ZF + AC + $\neg$CH

   (in fact, we may have $2^{\aleph_0} = \aleph_\alpha$ for α's that are arbitrarily large).

Chapter 8

Forcing

8.1 Theory

8.1.1 Ground Model

In [Kunen 80], six ways to view the forcing-setting are sketched. Here follows one of them; some of the others will be discussed later.

Inner models can only prove consistency of $\mathbf{V} = \mathbf{L}$ (and principles implied by $\mathbf{V} = \mathbf{L}$). Thus, if we want $\mathbf{V} \neq \mathbf{L}$, we should not *shrink* the universe, but *extend* it. Of course, with $\mathbf{V}$, this is impossible, since, by definition, it contains everything there is. The solution is to *postulate* a universe $\mathbf{M}$ that we may so extend.

Thus, ZFC($\mathbf{M}$) is the theory with non-logical symbols $\in$ and $\mathbf{M}$ (a new constant symbol) that has axioms

1. those of ZFC (including Foundation),
2. all sentences $\Phi^{\mathbf{M}}$ where Φ is a ZFC-axiom,
3. the statement "$\mathbf{M}$ is a non-empty, countable, transitive set".

The axioms of type 2 state that $(\mathbf{M}, \in)$ forms an *inner model* of ZFC in ZFC in the sense of Definition 7.5 (p. 53). (In what follows, often, the word "model" is used where "inner model" would be more accurate; cf. Exercise 199.)

Lemma 8.1 *If* ZFC *is consistent, then so is* ZFC($\mathbf{M}$).

Proof. If not, a contradiction would be derivable from 1., 3., and finitely many axioms $\Phi_1^{\mathbf{M}}, \dots, \Phi_k^{\mathbf{M}}$ of type 2. Thus,

$$\text{ZFC} \vdash [\mathbf{M} \text{ non-empty, countable, transitive } \rightarrow \neg(\Phi_1^{\mathbf{M}} \wedge \cdots \wedge \Phi_k^{\mathbf{M}})],$$

and hence (since $\mathbf{M}$ does not occur in the ZFC-axioms)

$$\text{ZFC} \vdash \forall m[m \text{ non-empty, countable, transitive } \rightarrow \neg(\Phi_1^m \wedge \cdots \wedge \Phi_k^m)].$$

However, we also have

$$\text{ZFC} \vdash \exists m[m \text{ non-empty, countable, transitive } \wedge \Phi_1^m \wedge \cdots \wedge \Phi_k^m]$$

—a contradiction. To see this, argue as follows (in ZFC): by Reflection, the Φ_i are true in some V_α; by Downward Löwenheim-Skolem, they are true in some countable subset of V_α, and by Collapsing, they are true in the transitive isomorph m of that subset. □

By the same argument, we could also consistently require $\mathbf{V} = \mathbf{L}$ in $\mathbf{M}$.

In ZFC($\mathbf{M}$), $\mathbf{M}$ is referred to as the *ground model.* Forcing is a method that produces, relative to ZFC($\mathbf{M}$), sets N such that

1. N is a (countable and) transitive extension of $\mathbf{M}$ s.t. $\mathrm{OR} \cap N = \mathrm{OR} \cap \mathbf{M}$,

2. all ZFC-axioms are true in N.

Depending on the details of the construction, it is possible to get, e.g., $2^{\aleph_0} \neq \aleph_1$ true in N. Using N for an interpretation of the ZF-language in ZFC($\mathbf{M}$) (the translation is $\Phi \mapsto \Phi^N$), it would then follow that the negation of CH is consistent with ZFC (provided ZFC is consistent). Note that all relative consistency results using forcing (as was the case with constructibility) employ only "finitary" (finite combinatorial) means.

Example 8.2 If we can produce $N \neq \mathbf{M}$, then $\mathbf{V} \neq \mathbf{L}$ is true in N and, hence, $\mathbf{V} \neq \mathbf{L}$ is consistent with ZFC.
Proof: Suppose that $\mathrm{OR} \cap N = \mathrm{OR} \cap \mathbf{M} = \alpha$. By absoluteness of $\mathbf{L}$, $\mathbf{L}^N = \mathbf{L}^{\mathbf{M}} = \mathrm{L}_\alpha \subset \mathbf{M}$. Thus, if N properly extends $\mathbf{M}$, then $\mathbf{L}^N \neq N$.

Exercises

198 ♣ Assume that $\mathbf{V} = \mathbf{L}$ holds, and that N is a transitive model of ZF such that $\omega_1 \subset N$. Show that CH is true in N. (Thus, we need *countable* models if we want independence of CH.)

199 ♣ Show: if ZFC is consistent, then so is ZFC($\mathbf{M}$) plus the axiom "$\mathbf{M}$ is *not* a ZF-model" (cf. Remark 7.48 p. 71 for the definition of "ZF-model"). (Thus, this extension of ZFC($\mathbf{M}$) is "ω-inconsistent".)

8.1.2 Partial orderings and generic filters

Since $\mathbf{M}$ is a set, one can consider any object $G \notin \mathbf{M}$ and try to build a model $N \supset \mathbf{M} \cup \{G\}$. However, many objects G will turn out to be unsuitable. For instance, if we want $\mathrm{OR} \cap N = \mathrm{OR} \cap \mathbf{M}$, we'd better not add a well-ordering of ω of type $\mathrm{OR} \cap \mathbf{M}$. (Cf. Exercise 209.)

Forcing models N are completely determined, next to $\mathbf{M}$, by

1. a partial ordering (*poset*) $(P, \leqslant) \in \mathbf{M}$,

2. a so-called $\mathbf{M}$-*generic filter* $G \subset P$.

Given these objects, the resulting forcing extension N is usually denoted by $\mathbf{M}[G]$: this will be the least transitive ZF-"model" extending $\mathbf{M}$ that contains G and has the same ordinals as $\mathbf{M}$.

In fact, it turns out that it is only the partial ordering P that is crucial: in the boolean-valued models-view of forcing, the filters are absent (they are needed only if one requires standard, transitive models), and usually P is homogeneous in a way that makes all $\mathbf{M}[G]$ (for $G \subset P$ $\mathbf{M}$-generic) look more or less alike.

In the forcing theory, it is nowhere required that the partial ordering is antisymmetric but we'll nevertheless assume this. Neither is it required that there is a greatest element, but there are a few advantages in having one.

Thus, let $(P, \leqslant, 1)$ be a partial ordering in $\mathbf{M}$ with greatest element 1. The elements of P are referred to as *conditions*, and, if $p \leqslant q$, we shall say that p *extends* or *refines* q. The idea is that conditions carry certain information about the new model to be constructed, and $p \leqslant q$ means that p carries as least as much information as q.
Warning. In some of the literature, the ordering $\leqslant$ is reversed, and $p \leqslant q$ means that q refines p. The usage here is derived from the boolean-valued approach.

Definition 8.3 A subset $G \subset P$ is a *filter* if

G1. $1 \in G$,

G2. $p \in G \wedge p \leqslant q \Rightarrow q \in G$,

G3. $p, q \in G \Rightarrow \exists r \in G(r \leqslant p, q)$.

A subset $D \subset P$ is *dense* if $\forall p \in P \exists q \in D(q \leqslant p)$: every condition has a refinement in D. Now a filter $G \subset P$ is $\mathbf{M}$-*generic* if

G4. whenever $D \subset P$ is dense and $D \in \mathbf{M}$, then $G \cap D \neq \emptyset$.

The following result is the *only* place where countability of $\mathbf{M}$ is needed.

Lemma 8.4 (Existence) *Every condition is element of an* $\mathbf{M}$*-generic filter.*

Proof. Assume $p \in P$. Since $\mathbf{M}$ is countable, we can enumerate the dense sets in $\mathbf{M}$: $D_0, D_1, D_2, \ldots$ Recursively, choose $p \geqslant p_0 \geqslant p_1 \geqslant p_2 \geqslant \cdots$ such that $p_i \in D_i$. Then $G =_{\text{def}} \{q \mid \exists n(p_n \leqslant q)\}$ is an $\mathbf{M}$-generic filter. □

Example 8.5 Suppose that $A, B \in \mathbf{M}$, where A is infinite and $|B| \geqslant 2$. (Simplest example: $A = \omega$, $B = 2$.) Let $P =_{\text{def}} \text{Fn}(A, B) =_{\text{def}} \{p \subset A \times B \mid p \text{ is a finite function s.t. } \text{Dom}(p) \subset A \wedge \text{Ran}(p) \subset B\}$. $p \leqslant q \equiv_{\text{def}} q \subset p$; $1 = \emptyset$.

Fix some $\mathbf{M}$-generic filter $G \subset P$ and define $f =_{\text{def}} \bigcup G$.

1. f is a function : $A \to B$.

   For suppose that $(a, b), (a, c) \in f$; say, $(a, b) \in p \in G$ and $(a, c) \in q \in G$. By G3, p and q have a common refinement r (in G). Then $(a, b), (a, c) \in r$, and $b = c$.

2. $\text{Dom}(f) = A$.

   For, let $a \in A$. $D =_{\text{def}} \{p \in P \mid a \in \text{Dom}(p)\}$ is dense and $\in \mathbf{M}$. (Why?) By G4, choose $p \in D \cap G$. Then $a \in \text{Dom}(p) \subset \text{Dom}(f)$.

3. $\text{Ran}(f) = B$.

   See Exercise 200.

Thus, G has the form $G = \{p \in P \mid p \subset f\}$ for a certain surjection $f : A \to B$.

If $\mathbf{M}[G]$ is the corresponding *generic extension* of $\mathbf{M}$ (still to be defined), then we'll have $G \in \mathbf{M}[G]$. Hence, $f = \bigcup G \in \mathbf{M}[G]$. (Why?) Thus, $\mathbf{M}[G] \models (f \text{ is a surjection } : A \to B)$.
N.B.: for readability, we usually employ $\models$ instead of quantifier relativization. Usually, it is irrelevant whether the intention is to use formulas or formulas* (cf. Definition 7.42 p. 69).

Note that forcing may "destroy initials": for instance, take $A =_{\text{def}} \omega$ and $B =_{\text{def}} \omega_1^{\mathbf{M}}$ ("the ordinal $\mathbf{M}$ thinks is ω_1" —see Notation 8.6). So the (Π^1_1) notion of an initial may not be absolute w.r.t. forcing-extensions. This example also illustrates the necessity (in Lemma 8.4) of $\mathbf{M}$ to be countable: if it isn't, $\omega_1^{\mathbf{M}}$ need not be countable either, in which case there just are no surjections from ω onto $\omega_1^{\mathbf{M}}$.

4. $G, f \notin \mathbf{M}$.

As a result, for $A = \omega$, $B = 2$, the poset $\mathrm{Fn}(A, B)$ produces a model $\mathbf{M}[G]$ and a function $f \in \mathbf{M}[G] \cap 2^\omega$ such that $\mathbf{M}[G] \models f \notin \mathbf{L}$ and hence $\mathbf{M}[G] \models \wp(\omega) \not\subset \mathbf{L}$.

Notation 8.6 In Chapter 7, Notation 7.9 (p. 56) introduced the notation $K^{\mathbf{M}}$, the relativized version of a class K in $\mathbf{M}$: if $K = \{x \in \mathbf{V} \mid \Phi(x)\}$, then $K^{\mathbf{M}} = \{x \in \mathbf{M} \mid \Phi^{\mathbf{M}}(x)\}$.

There is a similar notation w.r.t. operations. If $F : \mathbf{V} \to \mathbf{V}$ has been defined by the ZF-formula $\Psi(x, y)$, i.e.: $F(x) = y \equiv_{\text{def}} \Psi(x, y)$ (of course, this presupposes that $\forall x \exists ! y \Psi(x, y)$ is true), and we have that $(\forall x \exists y ! \Psi(x, y))^{\mathbf{M}}$, then Ψ defines an operation in $\mathbf{M}$, denoted by $F^{\mathbf{M}}$, and defined by $(x, y \in \mathbf{M})$ $F^{\mathbf{M}}(x) = y \equiv_{\text{def}} \Psi^{\mathbf{M}}(x, y)$.

200 ♣ Exercise. Prove claims 3 and 4 of Example 8.5.
Hints. Claim 3 uses that A is infinite. For claim 4: $D =_{\text{def}} \{p \in P \mid \neg(p \subset f)\} = P - G$ is a dense set in $\mathbf{M}$.

Definition 8.7 1. $p, q \in P$ are *compatible*, notation: $p \sim q$, if $\exists r \in P(r \leqslant p, q)$.

2. $p, q \in P$ are *incompatible*, notation: $p \perp q$, if $p \not\sim q$.

3. $Q \subset P$ is *(pairwise) compatible* if $\forall p, q \in Q(p \sim q)$.

4. $A \subset P$ is *(pairwise) incompatible* or an *antichain* if $\forall p, q \in A(p \neq q \Rightarrow p \perp q)$.

5. $E \subset P$ is *dense below* p if $\forall q \leqslant p \exists r \leqslant q(r \in E)$.

6. $O \subset P$ is *open* if $\forall p \in O \forall q \leqslant p(q \in O)$.

Exercises

201 ♣ Show: every generic filter is maximally compatible. I.e.: if $G \subset P$ is $\mathbf{M}$-generic, $G \subset G' \subset P$, and G' compatible, then $G' = G$.
Hint. If $p \in G' - G$, consider $D =_{\text{def}} \{q \mid q \leqslant p\} \cup \{q \mid q \perp p\}$.

202 ♣ Is it true that every maximally compatible filter is $\mathbf{M}$-generic?

203 ♣ A is an antichain. Show: A is a maximal antichain iff $\forall p \exists q \in A(p \sim q)$.

204 ♣ Suppose that G is a filter. Show that the following conditions are pairwise equivalent:

1. G is $\mathbf{M}$-generic, that is: it intersects every dense set in $\mathbf{M}$,

2. G intersects every maximal antichain in $\mathbf{M}$,

3. G intersects every $E \in \mathbf{M}$ for which $\forall p \exists q \in E(p \sim q)$,

4. if $p \in G$, then G intersects every $E \in \mathbf{M}$ that is dense below p,
5. G intersects every dense open set in $\mathbf{M}$.

Hints. 1 $\Rightarrow$ 2: Consider $D =_{\text{def}} \{q \mid \exists p \in A(q \leqslant p)\}$. 1 $\Rightarrow$ 4: Consider $D =_{\text{def}} E \cup \{q \mid q \perp p\}$.

205 ♣ The notion of genericity doesn't change if we weaken, in Definition 8.3,

G3: $p, q \in G \;\Rightarrow\; \exists r \in G(r \leqslant p, q)$

to

G3': $p, q \in G \;\Rightarrow\; \exists r(r \leqslant p, q)$.

Hint. For $p, q \in G$, consider $D =_{\text{def}} \{r \mid r \perp p \;\vee\; r \perp q \;\vee\; (r \leqslant p, q)\}$.

206 ♣

1. Suppose that $\forall p \in P \exists q, r(q, r \leqslant p \wedge q \perp r)$. (Note that $\mathrm{Fn}(A, B)$ has this property if A is infinite and B has at least two elements.) Show that $G \notin \mathbf{M}$. (Cf. Example 8.5 p. 79.)
2. If $\forall q, r \leqslant p(q \sim r)$, then $\{q \mid q \sim p\}$ is an $\mathbf{M}$-generic filter in $\mathbf{M}$.

Hint for 1: consider $D =_{\text{def}} P - G$.

207 ♣ Consider $P = \mathrm{Fn}(\omega \times \lambda, 2)$, where $\lambda = \omega_2^{\mathbf{M}}$. Let $G \subset P$ be $\mathbf{M}$-generic. Then (as in Example 8.5) $f =_{\text{def}} \bigcup G : \omega \times \lambda \to 2$. Define $g : \lambda \to \wp(\omega)$ by $g(\xi) =_{\text{def}} \{n \in \omega \mid f(n, \xi) = 1\}$. Show that g is an injection. Thus, $\mathbf{M}[G] \models \lambda \leqslant_1 \wp(\omega)$.

(It now looks, as if we have $\mathbf{M}[G] \models \neg\mathrm{CH}$, but of course, we can draw that conclusion only, if we know that $\omega_2^{\mathbf{M}} = \omega_2^{\mathbf{M}[G]}$. See below.)

208 ♣ Suppose that M and N are transitive ZF-models such that $M \subset N$. Show that $\alpha \in \mathrm{OR} \cap M \;\Rightarrow\; \omega_\alpha^M \leqslant \omega_\alpha^N$.

209 ♣ Let $P = \mathrm{Fn}(\omega \times \omega, 2)$, G be $\mathbf{M}$-generic, $f = \bigcup G$, $nRm \equiv_{\text{def}} f(n, m) = 1$. Show that R is not asymmetric. (Neither is it irreflexive, nor transitive. In particular, R is not a linear ordering of ω, let alone a wellordering.)

8.1.3 Names and generic extensions

Given an $\mathbf{M}$-generic G, the construction of $\mathbf{M}[G]$ uses codes in $\mathbf{M}$ for elements of (the future) $\mathbf{M}[G]$.

Definition 8.8 The class $\mathbf{V}^P$ of (P-)*names* or *codes* is the least class satisfying the inclusion

$$\wp(\mathbf{V}^P \times P) \subset \mathbf{V}^P.$$

In particular, every relation between names and conditions is a name.

Relativizing this inductive definition to $\mathbf{M}$, we obtain the class $\mathbf{M}^P = \mathbf{M} \cap \mathbf{V}^P \subset \mathbf{M}$ of P-names in $\mathbf{M}$.

Some examples of names: $\emptyset$, $\{(\emptyset,1)\}$ $(1 \in P)$, $\{\{(\emptyset,1)\}\} \times P$. Many more examples follow below.

210 ♣ Exercise. Show (as claimed in 8.8) that $\mathbf{M}^P = \mathbf{M} \cap \mathbf{V}^P$.
Hint. $\mathbf{V}^P$ is the unique solution of the equation $\wp(X \times P) = X$.

An arbitrary set $G \subset P$ can be used to "decode" names (retrieve an object from a name), but in practice, we'll decode using generic filters only. Decoding is just the collapsing map (see Lemma 7.52 p. 73) on $(\mathbf{M}^P, \varepsilon)$, where $\sigma \, \varepsilon \, \tau \equiv_{\text{def}} \exists p \in G((\sigma,p) \in \tau)$; and $\mathbf{M}[G]$ is the transitive collapse:

Definition 8.9 Recursively define, for $\tau \in \mathbf{M}^P$: $\tau_G = \{\sigma_G \mid \exists p{\in}G(\sigma,p) \in \tau\}$.
$\tau \in \mathbf{M}^P$ is called *code* or *name* of τ_G; τ_G is the *interpretation* of τ (modulo G).
Next: $\mathbf{M}[G] = \{\tau_G \mid \tau \in \mathbf{M}^P\}$.

Examples: $\emptyset_G = \emptyset$, $\{(\emptyset,1)\}_G = \{\emptyset\}$ (if $1 \in G$), $(\{\{(\emptyset,1)\}\} \times P)_G = \{\{\emptyset\}\}$ (if $G \neq \emptyset$).
Note that decoding is absolute. I.e., if $\tau \in \mathbf{M}^P$ and $N \supset \mathbf{M} \cup \{G\}$ is a transitive ZF-model, then $(\tau_G)^N = \tau_G$.

211 ♣ Exercise. The recursion of 8.9 is meaningful for arbitrary elements $\tau \in \mathbf{M}$. Show that it defines the same operation on $\mathbf{M}^P$.

In general, an element of $\mathbf{M}[G]$ has many different names. E.g., if $p \notin G$ and $\tau \in \mathbf{M}^P$, then $\emptyset = \emptyset_G = \{(\tau,p)\}_G$.

Corollary 8.10 $\mathbf{M}[G]$ *is transitive.*

Now, note that every element of $\mathbf{M}$ has a name. For, define $^\vee : \mathbf{M} \to \mathbf{M}^P$ recursively by $a^\vee = \{(b^\vee,1) \mid b \in a\}$. Note that, indeed, $a^\vee \in \mathbf{M}^P$.
The interpretation $a^\vee_G$ of $a^\vee$ is a —independently of the filter G:

212 ♣ Exercise. Show that $(a^\vee)_G = a$.
Hint. $1 \in G$.

Corollary 8.11 $\mathbf{M} \subset \mathbf{M}[G]$.

Lemma 8.12 *If* $N \supset \mathbf{M} \cup \{G\}$ *is a transitive model of* ZF, *then* $\mathbf{M}[G] \subset N$.

Proof. If $\tau \in \mathbf{M}^P$, then $\tau_G = (\tau_G)^N \in N$. □

Thus, *if* all ZF-axioms hold in $\mathbf{M}[G]$, *then* it is the smallest transitive ZF-model $\supset \mathbf{M} \cup \{G\}$. Note that every element of $\mathbf{M}[G]$ is definable in $\mathbf{M}[G]$ using two objects: a name in $\mathbf{M}$, and G.

213 ♣ Exercise. If $G \in \mathbf{M}$, then $\mathbf{M}[G] = \mathbf{M}$.

214 ♣ Exercise. Define $^* : \mathbf{M} \to \mathbf{M}^P$ by $a^* = \{(b^*,p) \mid b \in a \wedge p \in P\}$. Show that $(a^*)_G = a$.

Definition 8.13 $\Gamma =_{\text{def}} \{(p^\vee,p) \mid p \in P\}$.

Lemma 8.14 *1.* $\Gamma \in \mathbf{M}^P$*,*

2. $\Gamma_G = G$*,*

3. $G \in \mathbf{M}[G]$*.*

215 ♣ Exercise. Suppose that $G \subset P$ is $\mathbf{M}$-generic. Show: if G is $\mathbf{M}[G]$-generic as well, then $G \in \mathbf{M}$. (You can use the same P twice, but not the same G.)

Lemma 8.15 *1.* $\rho(\tau_G) \leqslant \rho(\tau)$*.*

2. $a \in \mathbf{M}[G] \Rightarrow \rho(a) \in \mathbf{M}$*.*

3. $\mathrm{OR} \cap \mathbf{M}[G] = \mathrm{OR} \cap \mathbf{M}$*.*

Exercises

216 ♣ Prove Lemma 8.15.

217 ♣ Suppose that $\sigma, \tau \in \mathbf{M}^P$. Then $\pi = \{(\sigma, 1), (\tau, 1)\} \in \mathbf{M}^P$ and (if $1 \in G$) $\pi_G = \{\sigma_G, \tau_G\}$. Thus, $\mathbf{M}[G]$ satisfies Pairing.

218 ♣ Suppose that $\sigma, \tau \in \mathbf{M}^P$. Then $\pi = \sigma \cup \tau \in \mathbf{M}^P$ and $\pi_G = \sigma_G \cup \tau_G$.

219 ♣ Suppose that $\tau \in \mathbf{M}^P$. Then $\bigcup \mathrm{Dom}(\tau) \in \mathbf{M}^P$ and $\bigcup \tau_G \subset (\bigcup \mathrm{Dom}(\tau))_G$. Thus, if $a \in \mathbf{M}[G]$, then $\bigcup a$ is included in an element of $\mathbf{M}[G]$.

220 ♣ Suppose that $\tau \in \mathbf{M}^P$. Then $\pi = \{(\rho, r) \mid \exists(\sigma, p) \in \tau \exists q[(\rho, q) \in \sigma \wedge r \leqslant p, q]\} \in \mathbf{M}^P$ and (if G is a filter) $\pi_G = \bigcup \tau_G$. Thus, $\mathbf{M}[G]$ satisfies Sumsets.

More detailed information is needed before we can turn to the other ZF-axioms.

8.1.4 Generic expansions and forcing

In order to obtain (more) information about $\mathbf{M}[G]$, we need the notion of forcing. With that, questions about the mysterious model $\mathbf{M}[G]$ can be translated into questions about the known model $\mathbf{M}$.

In order to get a better understanding of the transition from $\mathbf{M}$ to $\mathbf{M}[G]$, we divide the transition in two steps: *first*, we pass from $\mathbf{M} = (\mathbf{M}, \in)$ to the model $(\mathbf{M}, \in, G)$, where G is viewed as an additional one-argument relation symbol; *next*, we consider the transition from $(\mathbf{M}, \in, G)$ to $\mathbf{M}[G] = (\mathbf{M}[G], \in)$. It is the first transition where forcing is crucial; the second one hinges on a definability question.

Consider the vocabulary appropriate for expansions $(\mathbf{M}, \in, G, a)_{a \in \mathbf{M}}$ of $\mathbf{M}$ that (next to equality and the binary relation symbol $\in$), has a unary relation symbol Γ (for generic filters G) and constant symbols $\underline{a} = a$ for all $a \in \mathbf{M}$.

Definition 8.16 The *forcing relation* $\Vdash$ is a relation between conditions and first-order sentences of the above vocabulary, defined as follows:

$$\begin{array}{rcl} p \Vdash \Gamma(a) & \equiv & a \in P \ \wedge \ \forall q \leqslant p(q \sim a) \ \ (\Leftrightarrow \ a \in P \ \wedge \ \{r \mid r \leqslant a\} \text{ is dense below } p) \\ p \Vdash a = b & \equiv & a = b \\ p \Vdash a \in b & \equiv & a \in b \\ p \Vdash \Phi \ \wedge \ \Psi & \equiv & p \Vdash \Phi \ \wedge \ p \Vdash \Psi \\ p \Vdash \neg\Phi & \equiv & \neg \exists q \leqslant p(q \Vdash \Phi) \\ p \Vdash \forall x \Phi(x) & \equiv & \forall a \in \mathbf{M}(p \Vdash \Phi(a)). \end{array}$$

Note that we've taken $\wedge$, $\neg$ and $\forall$ as primitive logical operations; the definition turns out to be sensitive for this choice.

In the finite-function orderings $\mathrm{Fn}(A, B)$ (where A is infinite and $|B| \geqslant 2$ —see Example 8.5 p. 79) we have that $\forall q \leqslant p(q \sim s)$ iff $p \leqslant s$; so, in this case, the first equivalent of 8.16 could be simplified. Orderings in which this holds are called *separative*.

Definition 8.16 looks much like a truth definition, but the condition for $\neg$ stands out. One of its effects is that no condition forces contradictory statements Φ and $\neg\Phi$.

There are two ways to read 8.16.

(i) Schematically. Then for every ("genuine") formula $\Phi = \Phi(x_1, \ldots, x_k)$ that has the extra symbol Γ (but no constant symbols) a relation $\Vdash_\Phi = \{(p, a_1, \ldots, a_k) \in P \times \mathbf{M}^k \mid p \Vdash \Phi(a_1, \ldots, a_k)\}$ is defined; the definition is by a recursion on the nr. of logical symbols in Φ following the rules of 8.16. This may well be our preferred way of reading.

(ii) As a definition *in* the theory ZFC($\mathbf{M}$), much as Definition 7.45 (p. 70), using an extended notion of formula* (cf. Definition 7.42 p. 69).

By the way, the definition can be extended in the obvious way to sentences that employ conjunctions $\bigwedge_{i \in I} \Phi_i$ where $\{\Phi_i \mid i \in I\} \in \mathbf{M}$, but this appears not to be very useful.

The important properties of forcing are the ones below that have names: the Definability Lemma 8.17, the Extension Lemma 8.18, the Truth Lemma 8.22, the Completeness Lemma 8.23 and the Consequence Lemma 8.24. These results will often be used in future without explicit mentioning.

Note that 8.16 does not refer to G. Thus, as indicated under (i) above:

Lemma 8.17 (Definability) *Every relation* $\{(p, a_1, \ldots, a_k) \in P \times \mathbf{M}^k \mid p \Vdash \Phi(a_1, \ldots, a_k)\}$ *(where* $\Phi = \Phi(x_1, \ldots, x_k)$ *is a genuine formula) has a definition in* $(\mathbf{M}, \in)$.

In particular, in $\mathbf{M}$, Separation holds w.r.t. formulas that refer to $\Vdash$.

221 ♣ Exercise. Instead of Definition 8.16, define $\Vdash$ by: $p \Vdash \Phi \equiv$ for every $\mathbf{M}$-generic filter $G \ni p$, $(\mathbf{M}, \in, G) \models \Phi$. (Cf. Lemma 8.23; the interpretation of the non-logical symbols is the obvious one.) Under this definition, prove all equivalences of 8.16, except the one for $\neg$.

222 ♣ Exercise. If Γ does not occur in Φ and $p \in P$, then $(\mathbf{M}, \in) \models \Phi \ \Leftrightarrow \ p \Vdash \Phi$. (Thus, $\Vdash$ includes the notion of $\mathbf{M}$-truth.)

Lemma 8.18 (Extension) *If* $p \Vdash \Phi$ *and* $q \leqslant p$, *then* $q \Vdash \Phi$.

223 ♣ Exercise. Prove this.
Hint. Straightforward induction w.r.t. Φ.

Corollary 8.19 $p \Vdash \Phi \;\Rightarrow\; p \Vdash \neg\neg\Phi$.

Lemma 8.20 *If* $p \nVdash \Phi$*, then* $\exists q \leqslant p(q \Vdash \neg\Phi)$.

Proof. Induction w.r.t. Φ. If $p \nVdash \Gamma(a)$, then *either* $a \notin P$, *or* $a \in P$ and $\neg\forall q \leqslant p(q \sim p)$. In the first case, $p \Vdash \neg\Gamma(a)$. In the second case $q \leqslant p$ exists s.t. $q \perp a$, and we have that $q \Vdash \neg\Gamma(a)$. If $p \nVdash \neg\Phi$, then $q \leqslant p$ exists s.t. $q \Vdash \Phi$. A fortiori, $\forall r \leqslant q \exists s \leqslant r(s \Vdash \Phi)$, i.e., $q \Vdash \neg\neg\Phi$. The other cases are straightforward. □

224 ♣ Exercise. Complete the proof of Lemma 8.20.

225 ♣ Exercise. Show: $p \Vdash \neg\neg\Phi \;\Leftrightarrow\; p \Vdash \Phi$.

Corollary 8.21 *If* $\forall a \in \mathbf{M} \exists p \in G(p \Vdash \Phi(a))$*, then* $\exists p \in G \forall a \in \mathbf{M}(p \Vdash \Phi(a))$ *(for* $\mathbf{M}$*-generic* G*).*

Proof. Consider $D = \{p \mid \forall a(p \Vdash \Phi(a))\} \cup \{p \mid \exists a(p \Vdash \neg\Phi(a))\}$. Note that $D \in \mathbf{M}$. Also, D is dense. (If $p \notin D$, then $p \notin \{p \mid \forall a(p \Vdash \Phi(a))\}$. Say, $p \nVdash \Phi(a)$. By Lemma 8.20, $q \leqslant p$ exists s.t. $q \Vdash \neg\Phi(a)$.) Thus, $D \cap G \neq \emptyset$. If $\{p \mid \forall a(p \Vdash \Phi(a))\} \cap G \neq \emptyset$ we're done; therefore, it suffices to show that $\{p \mid \exists a(p \Vdash \neg\Phi(a))\} \cap G = \emptyset$. Thus, suppose $p \in G$, $p \Vdash \neg\Phi(a)$. By hypothesis, $q \in G$ exists s.t. $q \Vdash \Phi(a)$. By G3, choose $r \leqslant p, q$ and apply the Extension Lemma and the $\neg$-clause of 8.16. □

Lemma 8.22 (Truth) *For* $\mathbf{M}$*-generic* G *and all sentences* Φ*:*

$$(\mathbf{M}, \in, G) \models \Phi \;\Leftrightarrow\; \exists p \in G(p \Vdash \Phi).$$

Proof. Induction w.r.t. Φ.
1. Note that $p \Vdash \Gamma(p)$ for all p. Thus, if $(\mathbf{M}, \in, G) \models \Gamma(a)$, i.e., $a \in G$, then $\exists p \in G(p \Vdash \Gamma(a))$. Conversely, assume $p \in G$, $p \Vdash \Gamma(a)$. Then $a \in P$ and $\forall q \leqslant p(q \sim a)$. Note that $D = \{q \mid q \leqslant a\} \cup \{q \mid q \perp p\}$ is dense and $\in \mathbf{M}$. (For, assume $r \in P$. If $r \notin D$, then $r \notin \{q \mid q \perp p\}$, hence $r \sim p$; say, $q \leqslant r, p$, hence $q \sim a$, say, $s \leqslant q, a$ and hence $s \in D$ and $s \leqslant r$.) Choose $q \in D \cap G$. Then $q \perp p$ is impossible since $p, q \in G$. Thus, $q \leqslant a$ and $a \in G$.
2. Other atomic cases are trivial.
3. For the conjunction-case, use G3.
4. By Corollary 8.21 (and IH): $(\mathbf{M}, \in, G) \models \forall x \Phi(x)$, iff $\forall a \in \mathbf{M} : (\mathbf{M}, \in, G) \models \Phi(a)$, iff $\forall a \in \mathbf{M} \exists p \in G : p \Vdash \Phi(a)$, iff $\exists p \in G \forall a \in \mathbf{M} : p \Vdash \Phi(a)$, iff $\exists p \in G : p \Vdash \forall x \Phi(x)$.
5. Finally: $(\mathbf{M}, \in, G) \models \neg\Phi$, iff $(\mathbf{M}, \in, G) \not\models \Phi$, iff $\neg\exists p \in G : p \Vdash \Phi$, iff $\exists p \in G : p \Vdash \neg\Phi$. In the last equivalence, $\Leftarrow$ is due to G3, and $\Rightarrow$ to the $\neg$-clause of the forcing-definition and the fact that $\{p \mid p \Vdash \Phi\} \cup \{p \mid p \Vdash \neg\Phi\}$ is dense and $\in \mathbf{M}$. □

Lemma 8.23 (Completeness) $p \Vdash \Phi$ *iff, for all* $\mathbf{M}$*-generic* $G \ni p$*:* $(\mathbf{M}, \in, G) \models \Phi$.

Proof. $\Rightarrow$: Immediate from Lemma 8.22. $\Leftarrow$: Suppose that $p \nVdash \Phi$. Then (Exercise 225) $p \nVdash \neg\neg\Phi$, and (8.16) some $q \leqslant p$ forces $\neg\Phi$. Choose $G \ni q$ $\mathbf{M}$-generic. Then $(\mathbf{M}, \in, G) \models \neg\Phi$, and, also, $p \in G$. □

One can take 8.23 as the definition of forcing (see Exercise 221) —but then $\mathbf{M}$-definability of forcing (Lemma 8.17) isn't obvious at all.

Corollary 8.24 (Logical Consequence) $p \Vdash \Phi \models \Psi \;\Rightarrow\; p \Vdash \Psi$.

Exercises

226 ♣ Prove Corollary 8.24.

227 ♣ Every filter G that satisfies the equivalence of Lemma 8.22 is $\mathbf{M}$-generic.
Hint. Suppose that $D \in \mathbf{M}$ is dense. Check that $1 \Vdash \neg\forall x \neg(x \in \underline{D} \wedge \Gamma(x))$.

Note that a generic expansion $(\mathbf{M}, \in, G)$ need not satisfy Separation w.r.t. formulas that contain Γ. For instance, $G = \{p \in P \mid (\mathbf{M}, \in, G) \models \Gamma(p)\}$ is usually not in $\mathbf{M}$.

228 ♣ Show that generic expansions $(\mathbf{M}, \in, G)$ satisfy Collection (with formulas containing Γ).
Hint. Suppose that $a \in \mathbf{M}$ and $(\mathbf{M}, \in, G) \models \forall x \in a \exists y \Phi(x, y)$. By Collection in $\mathbf{M}$ (note that $P \in \mathbf{M}$), choose $b \in \mathbf{M}$ s.t. $\forall x \in a \forall p \in P[\exists y(p \Vdash \Phi(x, y)) \Rightarrow \exists y \in b(p \Vdash \Phi(x, y))]$.

229 ♣ Call a filter G generic* if for all Φ there exists $p \in G$ s.t. $p \Vdash \Phi$ or $p \Vdash \neg\Phi$. ("G decides every sentence." This definition is close to Cohen's original notion; it requires the non-schematic reading of forcing explained under (ii) above.) Prove Corollary 8.21, Lemma 8.22 and Lemma 8.23 for generic*. Show that generic* = generic.

8.1.5 From expansions to extensions

This section concerns the transition from $(\mathbf{M}, \in, G)$ to $\mathbf{M}[G]$ where G is generic. The main point is the following definability result.

Lemma 8.25 *There are formulas $x \varepsilon y$ and $x \approx y$ (containing Γ and parameters P and $\leqslant$) such that, for $\mathbf{M}$-generic G and $\pi, \sigma \in \mathbf{M}^P$:*

1. $\pi_G \in \sigma_G \;\Leftrightarrow\; (\mathbf{M}, \in, G) \models \pi \,\varepsilon\, \sigma,$
2. $\pi_G = \sigma_G \;\Leftrightarrow\; (\mathbf{M}, \in, G) \models \pi \approx \sigma.$

This result states that the structure $(\mathbf{M}[G], \in)$, via the codes in $\mathbf{M}^P$, is embedded in the structure $(\mathbf{M}, \in, G)$.

The proof of Lemma 8.25 that now follows will give you not much insight in the situation; therefore, you may prefer to skip it and go to the place where Lemma 8.25 is actually used: Lemma 8.30 (p. 88). This section's goal is Theorem 8.32 that collects all basic facts concerning forcing and generic extensions that will ever be needed.

Definition 8.26 Using a simultaneous recursion, define in $\mathbf{M}$ two relations on $P \times \mathbf{M}^P \times \mathbf{M}^P$, denoted by $p \vdash \pi \in \sigma$ resp., $p \vdash \neg(\pi \subset \sigma)$:

1. $p \vdash \pi \in \sigma \;\equiv\; \exists(\tau, q) \in \sigma[p \leqslant q \;\wedge\; p \vdash \pi = \tau]$,
2. $p \vdash \neg(\pi \subset \sigma) \;\equiv\; \exists(\tau, q) \in \pi[p \leqslant q \;\wedge\; p \vdash \tau \notin \sigma]$.

In these equivalences, the following two shorthands are used:

3. $p \vdash \pi = \tau \;\equiv_{\text{def}}\; \neg\exists q \leqslant p(q \vdash \neg(\pi \subset \tau)) \;\wedge\; \neg\exists q \leqslant p(q \vdash \neg(\tau \subset \pi))$,
4. $p \vdash \tau \notin \sigma \;\equiv_{\text{def}}\; \neg\exists q \leqslant p(q \vdash \tau \in \sigma)$.

230 ♣ Exercise. Justify the recursion in Definition 8.26.

Lemma 8.27 *1.* $q \leqslant p \vdash \pi \in \sigma \;\Rightarrow\; q \vdash \pi \in \sigma$,

2. $q \leqslant p \vdash \neg(\pi \subset \sigma) \;\Rightarrow\; q \vdash \neg(\pi \subset \sigma)$.

231 ♣ Exercise. Prove this.
Hint. Straightforward simultaneous induction.

The following proves Lemma 8.25.

Lemma 8.28 *The formulas*
$\pi \,\varepsilon\, \sigma \;\equiv\; \exists p(\Gamma(p) \,\wedge\, p \vdash \pi \in \sigma)$
and
$\pi \approx \sigma \;\equiv\; \neg\exists p(\Gamma(p) \,\wedge\, p \vdash \neg(\pi \subset \sigma)) \,\wedge\, \neg\exists p(\Gamma(p) \,\wedge\, p \vdash \neg(\sigma \subset \pi))$
satisfy the equivalences of Lemma 8.25.

Proof. It clearly suffices to prove that

1. $\pi_G \in \sigma_G \;\Leftrightarrow\; \exists p \in G(p \vdash \pi \in \sigma)$, and
2. $\pi_G \not\subset \sigma_G \;\Leftrightarrow\; \exists p \in G(p \vdash \neg(\pi \subset \sigma))$.

To show this, note first that, by definition of the decoding map $\pi \mapsto \pi_G$:

(i) $\pi_G \in \sigma_G \;\Leftrightarrow\; \exists(\tau, q) \in \sigma[q \in G \,\wedge\, \neg(\pi_G \not\subset \tau_G) \,\wedge\, \neg(\tau_G \not\subset \pi_G)]$,

(ii) $\pi_G \not\subset \sigma_G \;\Leftrightarrow\; \exists(\tau, q) \in \pi[q \in G \,\wedge\, \neg(\tau_G \in \sigma_G)]$.

What is more, by induction, it follows that these equivalences *characterize* the relations on the left-hand sides. *Therefore*, it suffices to show that the conditions in the right-hand sides of 1. and 2. above also satisfy equivalences (i) and (ii), i.e.: it suffices to show

(i)' $\exists p \in G(p \vdash \pi \in \sigma) \;\Leftrightarrow$
$$\exists(\tau, q) \in \sigma[q \in G \,\wedge\, \neg\exists p \in G(p \vdash \neg(\pi \subset \tau)) \,\wedge\, \neg\exists p \in G(p \vdash \neg(\tau \subset \pi))],$$

(ii)' $\exists p \in G(p \vdash \neg(\pi \subset \sigma)) \;\Leftrightarrow\; \exists(\tau, q) \in \pi[q \in G \,\wedge\, \neg\exists p \in G(p \vdash \tau \in \sigma)]$.

For this, we're going to use the following equivalences:

(a) $\exists p \in G(p \vdash \pi = \tau) \;\Leftrightarrow\; \neg\exists p \in G(p \vdash \neg(\pi \subset \tau)) \,\wedge\, \neg\exists p \in G(p \vdash \neg(\tau \subset \pi))$,

(b) $\exists p \in G(p \vdash \tau \notin \sigma) \;\Leftrightarrow\; \neg\exists p \in G(p \vdash \tau \in \sigma)$.

Proof: As to (a) $\Rightarrow$: If $p \in G$, $p \vdash \pi = \tau$, and, say, $q \in G$, $q \vdash \neg(\pi \subset \tau)$, choose $r \leqslant p, q$ in G; then by Lemma 8.27, $r \vdash \neg(\pi \subset \tau)$, contradicting shorthand 8.26.3.
(b) $\Rightarrow$: Idem.
(b) $\Leftarrow$: Suppose that $\neg\exists p \in G(p \vdash \tau \in \sigma)$. By Lemma 8.22, let $q \in G$ force this, i.e.: $q \Vdash [\neg\exists p(\Gamma(p) \,\wedge\, p \vdash \tau \in \sigma)]$.
Claim. $q \vdash \tau \notin \sigma$.
Proof. By shorthand 8.26.4, we have to show $\neg\exists r \leqslant q(r \vdash \tau \in \sigma)$. Suppose that $r \leqslant q$ exists s.t. $r \vdash \tau \in \sigma$. Choose $G' \ni r$ **M**-generic. Then $q \in G'$. By choice of q, $(\mathbf{M}, \in, G') \models [\neg\exists p(\Gamma(p) \,\wedge\, p \vdash \tau \in \sigma)]$, i.e.: G' doesn't have an element that $\vdash \tau \in \sigma$, contradicting the

supposed existence of r.
(a)$\Leftarrow$: Similar. Assume the right-hand side of (a) and choose $q \in G$ that forces this. Then $q \vdash \pi = \tau$. (For instance, to see that —cf. shorthand 8.26.3— $\neg\exists r \leqslant q(r \vdash \neg(\pi \subset \tau))$, assume that $r \leqslant q$ is s.t. $r \vdash \neg(\pi \subset \tau)$. Choose $G' \ni r$ generic. Then $q \in G'$, and by choice of q: $(\mathbf{M}, \in, G') \models [\neg\exists p(\Gamma(p) \wedge p \vdash \neg(\pi \subset \tau))]$: contradicting existence of r.

Now, note that the RHS's of (a) and (b) are parts of the RHS's of (i)' and (ii)'. Replacing the latter parts by the LHS's of (a) and (b) transforms (i)' and (ii)' into:

(i)" $\exists p \in G(p \vdash \pi \in \sigma) \Leftrightarrow \exists(\tau, q) \in \sigma[q \in G \wedge \exists p \in G(p \vdash \pi = \tau)]$,

(ii)" $\exists p \in G(p \vdash \neg(\pi \subset \sigma)) \Leftrightarrow \exists(\tau, q) \in \pi[q \in G \wedge \exists p \in G(p \vdash \tau \notin \sigma)]$.

But, these equivalences follow immediately from Definition 8.26 and the filter properties of G. □

Lemma 8.25 (p. 86) proved, we use it to translate formulas, thereby making the transition from expansions $(\mathbf{M}, \in, G)$ to extensions $(\mathbf{M}[G], \in)$, (defining the latter in the former).

Definition 8.29 The translation $\Phi \mapsto \Phi^*$ from ZF-formulas Φ (using $\in$ only) to formulas Φ^* that also use the relation symbol Γ (and the parameter P)

(i) replaces, in Φ, atomic formulas $x \in y$ resp., $x = y$ by $x \,\varepsilon\, y$ resp., $x \approx y$, and

(ii) relativizes quantifiers to $\mathbf{M}^P$.

Lemma 8.30 *If* $\Phi = \Phi(x_1, \ldots, x_k)$ *is a* ZF*-formula,* $\tau_1, \ldots, t_k \in \mathbf{M}^P$*, and* G *is* $\mathbf{M}$*-generic, then*

$$(\mathbf{M}[G], \in) \models \Phi((\tau_1)_G, \ldots, (\tau_k)_G) \Leftrightarrow (\mathbf{M}, \in, G) \models \Phi^*(\tau_1, \ldots, \tau_k).$$

Proof. Induction w.r.t. Φ. For atomic formulas, this is the content of Lemma 8.25 (p. 86). The induction steps are trivial. □

We now adjust the forcing relation using *, in order that it applies to generic extensions $(\mathbf{M}[G], \in)$ instead of expansions $(\mathbf{M}, \in, G)$:

Definition 8.31

$$p \Vdash^* \Phi(\tau_1, \ldots, \tau_k) \equiv_{\text{def}} p \Vdash \Phi^*(\tau_1, \ldots, \tau_k).$$

The following theorem collects all basic facts about forcing and generic extensions.

Theorem 8.32

1. Truth Lemma.
   If G *is* $\mathbf{M}$*-generic, then* $\mathbf{M}[G] \models \Phi((\tau_1)_G, \ldots, (\tau_k)_G) \Leftrightarrow \exists p \in G(p \Vdash^* \Phi(\tau_1, \ldots, \tau_k))$.
2. Extension Lemma.
   $q \leqslant p \Vdash^* \Phi \Rightarrow q \Vdash^* \Phi$.
3. Definability Lemma.
   Every relation $\{(p, \pi_1, \ldots, \pi_k) \mid p \Vdash^* \Phi(\pi_1, \ldots, \pi_k)\}$ *is definable over* $\mathbf{M}$.

4. Completeness Lemma.

$p \Vdash^* \Phi(\tau_1, \dots \tau_k)$ *holds iff, for all* $\mathbf{M}$*-generic* $G \ni p$*:* $\mathbf{M}[G] \models \Phi((\tau_1)_G, \dots, (\tau_k)_G)$.

5. Consequence Lemma.

$p \Vdash^* \Phi \models \Psi \;\Rightarrow\; p \Vdash^* \Psi$.

6. Behavior w.r.t. logical operations.

$$\begin{aligned} p \Vdash^* \Phi \wedge \Psi &\equiv p \Vdash^* \Phi \wedge p \Vdash^* \Psi \\ p \Vdash^* \neg\Phi &\equiv \neg\exists q \leqslant p(q \Vdash^* \Phi) \\ p \Vdash^* \forall x \Phi(x) &\equiv \forall \pi \in \mathbf{M}^P (p \Vdash^* \Phi(\pi)). \end{aligned}$$

Proof. Straightforward from the appropriate results in Subsection 8.1.4 and Lemma 8.30. □

From now on, $\Vdash^*$ is the only forcing relation that we'll use. Therefore, in the sequel, we drop the * and write $\Vdash$ for $\Vdash^*$.

Exercises

232 ♣ To the ZF-language, add the unary relation symbol $\mathbf{M}$ that, in models $(\mathbf{M}[G], \in)$, will be interpreted by $\mathbf{M}$. Extend the translation * suitably: replace atoms $\mathbf{M}(x)$ by the formula $\exists y(x \approx y^\vee)$. Show that 8.30 and 8.32 remain valid under this extension. Thus, everything works for generic extensions $(\mathbf{M}[G], \in, \mathbf{M})$.

233 ♣ Show that, for $a \in \mathbf{M}$: $p \Vdash \forall x \in a^\vee \Phi(x) \;\Leftrightarrow\; \forall b \in a(p \Vdash \Phi(b^\vee))$.

8.1.6 $\mathbf{M}[G] \models$ ZFC

Lemma 8.33 *Suppose that* $\mathbf{M}[G] \models \forall x(\Phi(x, \pi_G) \to x \in \sigma_G)$*. Let* $\tau =_{def} \{(\mu, p) \in \mathrm{Dom}(\sigma) \times P \mid p \Vdash \Phi(\mu, \pi)\}$*. Then* $\mathbf{M}[G] \models \forall x(\Phi(x, \pi_G) \leftrightarrow x \in \tau_G)$*.*

Proof. $(\to)$ Assume that $\mathbf{M}[G] \models \Phi(a, \pi_G)$. By assumption, $a \in \sigma_G$; say, $a = \mu_G$ and $\mu \in \mathrm{Dom}(\sigma)$. Choose $p \in G$ s.t. $p \Vdash \Phi(\mu, \pi)$. Then $(\mu, p) \in \tau$ and $\mu_G \in \tau_G$.
$(\leftarrow)$ Assume $a \in \tau_G$. Say, $a = \mu_G$, $(\mu, p) \in \tau$, $p \in G$. Then $p \Vdash \Phi(\mu, \pi)$ and hence $\mathbf{M}[G] \models \Phi(\mu_G, \pi_G)$. □

Corollary 8.34 $\mathbf{M}[G]$ *satisfies Separation.*

N.B.: From Exercise 232 it follows that Separation is satisfied in $(\mathbf{M}[G], \in, \mathbf{M})$ as well.

Corollary 8.35 *Every subset of* σ_G *in* $\mathbf{M}[G]$ *has a name* $\subset \mathrm{Dom}(\sigma) \times P$ *in* $\mathbf{M}^P$*.*

Proof. Suppose that $\pi_G \subset \sigma_G$. Thus, $\mathbf{M}[G] \models a \in \pi_G \to a \in \sigma_G$. By 8.33, $\tau =_{\mathrm{def}} \{(\mu, p) \in \mathrm{Dom}(\sigma) \times P \mid p \Vdash \mu \in \pi\}$ is a name for π_G. □

Corollary 8.36 $\mathbf{M}[G]$ *satisfies Powersets.*

Proof. By 8.35, for $\tau =_{\mathrm{def}} (\mathbf{M} \cap \wp(\mathrm{Dom}(\sigma) \times P)) \times \{1\}$ we have that $\mathbf{M}[G] \cap \wp(\sigma_G) \subset \tau_G$. Use Separation. □

Lemma 8.37 $\mathbf{M}[G]$ *satisfies Sumsets.*

Proof. Let $\tau =_{\text{def}} \bigcup_{\mu \in \text{Dom}(\sigma)}(\text{Dom}(\mu) \times \{1\})$. Then $\bigcup \sigma_G \subset \tau_G$: Suppose $a \in \sigma_G$. Say, $a = \mu_G$, $\mu \in \text{Dom}(\sigma)$. Then $\mu_G \subset (\text{Dom}(\mu) \times \{1\})_G \subset \tau_G$. □

Lemma 8.38 $\mathbf{M}[G]$ *satisfies Collection.*

N.B.: By Exercise 232, Collection also holds for models $(\mathbf{M}[G], \in, \mathbf{M})$.

Proof. Suppose $\mathbf{M}[G] \models \forall a \in \pi_G \exists b \Phi(a, b)$. By Collection in $\mathbf{M}$, choose $u \in \mathbf{M}$, $u \subset \mathbf{M}^P$, such that $\forall p \in P \forall \sigma \in \text{Dom}(\pi)[\exists \mu (p \Vdash \Phi(\sigma, \mu)) \;\Rightarrow\; \exists \mu \in u(p \Vdash \Phi(\sigma, \mu))]$. (This uses that P is a *set in* $\mathbf{M}$.) Let $\tau =_{\text{def}} u \times \{1\}$. Then $\mathbf{M}[G] \models \forall a \in \pi_G \exists b \in \tau_G \Phi(a, b)$. □

Lemma 8.39 $\mathbf{M}[G]$ *satisfies* AC.

Proof. Suppose $\pi_G \in \mathbf{M}[G]$, $\pi \in \mathbf{M}^P$. We have that $\text{Dom}(\pi) \in \mathbf{M} \models \text{AC}$; thus, we may choose an ordinal $\alpha \in \text{OR} \cap \mathbf{M}$ and a surjection $f : \alpha \to \text{Dom}(\pi)$, $f \in \mathbf{M}$. Then also $f \in \mathbf{M}[G]$. In $\mathbf{M}[G]$, define the map $h : \alpha \to \{\sigma_G \mid \sigma \in \text{Dom}(\pi)\}$ by $h(\xi) = (f(\xi))_G$. Clearly, h is a surjection from α onto $\{\sigma_G \mid \sigma \in \text{Dom}(\pi)\}$. But, $\pi_G \subset \{\sigma_G \mid \sigma \in \text{Dom}(\pi)\}$. Thus, π_G has a well-ordering in $\mathbf{M}[G]$. □

8.1.7 Epilogue

History

According to [Dawson 98], Gödel has tried to settle the independence of AC and GCH during the years 1940-42. Cohen (a Stanford analyst that asked the foundations people there about their most famous unsolved problem) discovered forcing in 1963. His ground model was $\mathbf{M} = \text{L}_\alpha$ for a suitable countable α; forcing and $\mathbf{M}[G]$ were defined using a complicated ("ramified") language derived from the constructibility methodology. (See [Cohen 65], [Cohen 66]. A first stumbling block was that, at the time, no detailed treatment of constructibility à la Gödel's 1938-note —such as the one of Chapter 7— was available.)

What Cohen defined was "strong forcing" $\Vdash^s$ with intuitionistic properties. For a classical context, it is the "weak forcing" defined here that is relevant. The connection is, that $p \Vdash \Phi \;\Leftrightarrow\; p \Vdash^s \neg\neg\Phi$ (cf. the corresponding translation of intuitionistic logic into classical logic). In Cohen's setting, conditions are (finite) sets of atomic facts about the generic object. The $\neg$-case was avoided using prenex forms. Later, Scott discovered the present $\neg$-clause.

Solovay and Scott discovered that weak forcing amounts to evaluating formulas (using the map $\Phi \mapsto \{p \mid p \Vdash \Phi\}$) in a complete boolean algebra consisting of regular open sets (in the order-topology) of conditions. (This explains the curious clauses for Γ and $\neg$.) They developed boolean-valued models (cf. [Bell 77]).

This set-up was translated back to forcing and generic models by [Shoenfield 71] (the preprint was available at the congress in 1967): nowadays, this is the usual way to treat forcing, also used by [Kunen 80]. The above treatment interpolates expansions $(\mathbf{M}, \in, G)$ into the transition from $\mathbf{M}$ to $\mathbf{M}[G]$, thus separating "model-theoretic" forcing from a definability result. Without this, the two collapse to "set-theoretic forcing" with the consequential cumbersome atomic cases.

Boolean-valued models

A set A in a topological space X is *regular open* if $A = A^{coco}$ $(= A^{-o})$ (where ^{c} denotes complementation, ^{o} interior, $^-$ closure).

Fact: the regular-opens form a complete boolean algebra $\mathrm{RO}(X)$ ([Halmos] pp.13–25) under the definitions $0 = \emptyset$, $1 = X$, $A \wedge B = A \cap B$, $A' = A^{co}$, $\bigwedge_i A_i = (\bigcap_i A_i)^{coco}$.

In the left-open order topology on a partial ordering P we have that, for $A \subset P$: $p \in A^{co} \Leftrightarrow \forall q \leqslant p(q \in A^c) \Leftrightarrow \forall q \leqslant p(q \notin A)$; hence $p \in A^{coco} \Leftrightarrow \forall q \leqslant p \exists r \leqslant q(r \in A)$. It follows that the map $\Phi \mapsto \{p \mid p \Vdash \Phi\}$ leads to $\mathrm{RO}(P)$ and transforms logical operations into the boolean ones.

Note that P is separative iff every open is also regular-open.

If P is a partial ordering, we can associate with it two topological spaces and their regular-open algebras: (i) P itself, with basic-opens $\{q \mid q \leqslant p\}$; the regular-open algebra of this space is denoted by $\mathrm{RO}(P)$; (ii) the space $\{G \subset P \mid G \text{ is a maximal filter}\}$ with basic opens $N(p) = \{G \mid p \in G\}$. *Fact:* the two algebras are isomorphic. ([Takeuti-Zaring] Theorem 5.18.) We have that $p \Vdash \Phi$ iff, $(N(p) - \{G \mid G \text{ is } \mathbf{M}\text{-generic} \wedge \mathbf{M}[G] \models \Phi\})$ is meagre (Ryll-Nardzewski). (There is a correspondence between Lemma 8.4 p. 79 and the Baire category theorem.) Note: for $P = \mathrm{Fn}(A, B)$, maximal filters have the form $\{p \mid p \subset f\}$ with $f : A \to B$ and the space under (ii) is the product space B^A where B is discrete.

Other settings

Put $\mathbf{M} = \mathbf{V}$ (!) and choose $P \in \mathbf{V}$. Define $\Vdash$. Clearly, the condition $1 \in P$ forces that Γ is generic (cf. Exercise 227 p. 86). This shows that the theory ZFC(Γ), with axioms: ZFC plus the statement "Γ is $\mathbf{V}$-generic" is consistent (as long as we don't allow Γ in Separation).

This theory describes the models $(\mathbf{M}, \in, G)$. The translation $\Phi \mapsto \exists p \in G(p \Vdash \Phi^*)$ interprets ZFC into ZFC, and, depending on the choice of P, validates extra hypothese such as $\neg$CH. This is the setting of [Shoenfield 67] and the Czech semi-set theory. Advantage: no separate $\mathbf{M}$, but the new universe $\mathbf{M}[G] = \mathbf{V}[G]$ (statements Φ^* in a sense refer to this) remains diffuse.

And, in fact, one can do without generic objects altogether and translate $\Phi \mapsto 1 \Vdash \Phi^*$ —but this amounts to the boolean-valued approach.

Letting $\mathbf{M} = \mathbf{V}$ and considering "imaginary" extensions $\mathbf{V}[G]$ is the viewpoint in the nice treatment [Baumgartner] (which also stresses that all applications of forcing boil down to producing partial orderings with specific combinatorial properties). This approach is often encountered in the litterature, but we shall usually be more cautious.

Other references: [Burgess 77] is a fast way into forcing, dealing only with applications; [Jech 78] mixes forcing and boolean-valued models and presents a host of applications.

8.2 Consistency of $\mathbf{V} \neq \mathbf{L}$

234 ♣ Exercise. Show that $\mathrm{Fn}(\omega, 2)$ forces that $\wp(\omega) \not\subset \mathbf{L}$.
Hint. See Example 8.5 (p. 79).

235 ♣ Exercise. Show that $\mathrm{Fn}(\omega, \omega_1)$ forces that $\mathbf{L} \cap \wp(\omega)$ is countable.

8.3 Alternatives to CH

Suppose that λ is an initial of $\mathbf{M}$, that $P = \mathrm{Fn}(\omega \times \lambda, 2)$ (see Example 8.5 p. 79), and that $G \subset P$ is $\mathbf{M}$-generic. Then $f =_{\mathrm{def}} \bigcup G : \omega \times \lambda \to 2$ is a function in $\mathbf{M}[G]$ and $\xi \mapsto a_\xi =_{\mathrm{def}} \{n \in \omega \mid f(n, \xi) = 1\}$ is a function: $\lambda \to \wp(\omega)$ in $\mathbf{M}[G]$. As in Example 8.5, $\mathrm{Dom}(f) = \omega \times \lambda$, and the map $\xi \mapsto a_\xi$ is injective. Thus, in $\mathbf{M}[G]$, we have that $\lambda \leqslant_1 \wp(\omega)$.

Suppose now that, for instance, $\lambda = \omega_3^{\mathbf{M}}$. Then *if we would know that, also* $\lambda = \omega_3^{\mathbf{M}[G]}$, it would follow in $\mathbf{M}[G]$ that $\omega_3 \leqslant_1 \wp(\omega)$, i.e.: $2^{\aleph_0} \geqslant \aleph_3$. This problem of preserving initials (see 8.40) motivates most of the present section.

236 ♣ Exercise. If $\mathbf{M} \subset N$, N a transitive model of ZF, then:

1. every initial of N in $\mathbf{M}$ is also an initial of $\mathbf{M}$,
2. every regular initial of N in $\mathbf{M}$ is also a regular initial of $\mathbf{M}$,
3. if λ is an initial of N in $\mathbf{M}$, then $\mathrm{cf}^N(\lambda) \leqslant \mathrm{cf}^{\mathbf{M}}(\lambda)$.

Definition 8.40 The partial ordering P *preserves*

1. *initials* if, for every $\mathbf{M}$-generic G: every initial of $\mathbf{M}$ is also an initial of $\mathbf{M}[G]$,
2. *regularity* if, for every generic G: every regular initial of $\mathbf{M}$ is also a regular initial of $\mathbf{M}[G]$,
3. *cofinalities* if, for every generic G and initial λ of $\mathbf{M}$: $\mathrm{cf}^{\mathbf{M}[G]}(\lambda) = \mathrm{cf}^{\mathbf{M}}(\lambda)$.

Lemma 8.41 *If P preserves regularity, then it also preserves initials and cofinalities.*

Proof. *Cofinalities:*
Assume that α is the cofinality of the ordinal λ in $\mathbf{M}$. Then a function $h \in \mathbf{M}$ exists such that the statement

[A] "$h : \alpha \to \lambda$ is order-preserving and has $\mathrm{Ran}(h)$ unbounded in λ"

is true in $\mathbf{M}$. Since [A] can be written with restricted quantifiers, it is true in $\mathbf{M}[G]$ as well. But then we have that the sentence

[B] $\mathrm{cf}(\alpha) = \mathrm{cf}(\lambda)$

is also true in $\mathbf{M}[G]$, as the implication [A] $\Rightarrow$ [B] is a theorem of ZF. (See Exercise 122 p. 49.) However, α is regular in $\mathbf{M}$ and stays that way in $\mathbf{M}[G]$ by hypothesis. Thus, we have that $\alpha = \mathrm{cf}(\lambda)$ in $\mathbf{M}[G]$.
Initials:
Suppose that λ is an initial in $\mathbf{M}$. If λ happens to be regular in $\mathbf{M}$, it stays that way in $\mathbf{M}[G]$ and so it also is an initial in $\mathbf{M}[G]$. If λ is singular in $\mathbf{M}$, then (since by AC, successor-initials are regular; see Lemma 6.34 p. 49) $\lambda = \bigcup\{\alpha < \lambda \mid \alpha \text{ is regular in } \mathbf{M}\} = \bigcup\{\alpha < \lambda \mid \alpha \text{ is regular in } \mathbf{M}[G]\}$ and so λ is an initial of $\mathbf{M}[G]$ in this case as well. □

Definition 8.42 P is *ccc* (satisfies the *countable chain condition*) if every antichain (set of pairwise incompatible conditions) is countable.

Of course, every partial ordering in $\mathbf{M}$ is ccc, since $\mathbf{M}$ is countable. The crucial property, however, is to be ccc *in* $\mathbf{M}$.

Here is the first result illustrating that forcing applications boil down to infinitary combinatorics (which is, according to [Kunen 80], the field "that used to be called set theory before there were independence proofs").

Theorem 8.43 *If (in* $\mathbf{M}$*)* P *is ccc, then* P *preserves regularity (and initials, and cofinalities).*

Proof. Assume that λ is regular in $\mathbf{M}$. We may as well assume that $\omega < \lambda$, since ω is preserved (and stays regular) anyway. Suppose that λ is *not* regular in $\mathbf{M}[G]$. Thus, $\alpha < \lambda$ and a function $h : \alpha \to \lambda$ in $\mathbf{M}[G]$ exist such that $\mathrm{Ran}(h)$ is unbounded in λ. Choose a name π for h. By the Truth lemma, some $p \in G$ forces that $\pi : \alpha^\vee \to \lambda^\vee$ is a function with $\mathrm{Ran}(\pi)$ unbounded in $\lambda^\vee$.

With slightly more effort, the rest of the argument could take place entirely in $\mathbf{M}$. Define, for $\xi < \alpha$: $a_\xi =_{\mathrm{def}} \{\delta < \lambda \mid \exists q \leqslant p(q \Vdash \pi(\xi^\vee) = \delta^\vee)\}$. (Intuition: a_ξ is the set of "possible function-values" δ such that for some generic $G' \ni p$, $\pi_{G'}(\xi) = \delta$ — but *this* definition transcends $\mathbf{M}$.)

Claim. Every a_ξ is countable$^{\mathbf{M}}$.
Proof. For every $\delta \in a_\xi$, pick $q_\delta \leqslant p$ that forces $\pi(\xi^\vee) = \delta^\vee$. Then $A =_{\mathrm{def}} \{q_\delta \mid \delta \in a_\xi\}$ is an antichain: if $q \leqslant q_{\delta_1}, q_{\delta_2}$, then q forces $\delta_1^\vee = \pi(\xi^\vee) = \delta_2^\vee$, and so $\delta_1 = \delta_2$ (this holds in every $\mathbf{M}[G']$ for which $q \in G'$). The Claim follows by P being ccc$^{\mathbf{M}}$.

In $\mathbf{M}$, $a =_{\mathrm{def}} \bigcup_{\xi<\alpha} a_\xi \leqslant_1 \alpha \times \omega =_1 \alpha$, from which it follows that, since λ is regular in $\mathbf{M}$, a must be bounded in λ; say, by $\delta < \lambda$. Pick $\xi \in \alpha$ such that $\delta < h(\xi)$. By the Truth Lemma, $r \in G$ exists forcing $\pi(\xi^\vee) = (h(\xi))^\vee$. Finally, choosing $q \leqslant p, r$ in G, we see that $h(\xi) \in a_\xi$, a contradiction. $\square$

Thus, it follows from the following lemma that we can invalidate CH (and in fact, can get $2^{\aleph_0}$ arbitrarily large) by means of forcing:

Lemma 8.44 *If* J *is countable (and* I *arbitrary), then* $\mathrm{Fn}(I, J)$ *is ccc.*

In turn, this result follows from the Δ*-systems lemma.*

Definition 8.45 A collection of sets A is a Δ*-system with root* w if, for all $a, a' \in A$ s.t. $a \neq a'$: $a \cap a' = w$.

Lemma 8.46 (Δ-systems lemma) *Every uncountable collection of finite sets includes an uncountable* Δ*-system.*

Proof of Lemma 8.44 from Lemma 8.46.
Suppose that $A \subset \mathrm{Fn}(I, J)$ is an uncountable antichain. First, suppose that $\{\mathrm{Dom}(p) \mid p \in A\}$ is *countable.* Then (by a pigeon-hole principle) some $w \subset I$ exists s.t. $\{p \in A \mid \mathrm{Dom}(p) = w\}$ is uncountable. However, $\{p \in A \mid \mathrm{Dom}(p) = w\} \subset J^w$, and, since w is finite, J^w must be countable. It follows, that $\{\mathrm{Dom}(p) \mid p \in A\}$ must be *uncountable.* By the Δ-systems lemma, we can find an uncountable Δ-system $\Delta \subset \{\mathrm{Dom}(p) \mid p \in A\}$ with a root w. For every $D \in \Delta$, choose $p_D \in A$ s.t. $\mathrm{Dom}(p_D) = D$. Consider $B =_{\mathrm{def}} \{p_D \mid D \in \Delta\}$. Clearly, if $p, q \in B$ and $p \neq q$, then $(p|w) \perp (q|w)$; therefore, $\{p|w \mid p \in B\}$ is an uncountable antichain. But (again), $\{p|w \mid p \in B\} \subset J^w$ and J^w is only countable. $\square$

Proof of Lemma 8.46.
Suppose that A is an uncountable collection of finite sets. Put $A_n =_{\text{def}} \{a \in A \mid |a| = n\}$; then $A = \bigcup_n A_n$. By a pigeon-hole principle, it follows that at least one of the A_n must be uncountable. Therefore, it suffices to show that every uncountable collection A of n-element sets contains an uncountable Δ-system. The proof uses induction w.r.t. n.
$n = 0$.
This case is trivial, since there is but one 0-element set.
$n + 1$.
Distinguish two cases.
(i) For some x, $B =_{\text{def}} \{a \in A \mid x \in a\}$ is uncountable. In that case, by IH, there is an uncountable Δ-system $C \subset \{a - \{x\} \mid a \in B\}$ with some root w, and the corresponding $\{a \mid a - \{x\} \in C\}$ is the Δ-system with root $w \cup \{x\}$ as required.
(ii) There is no such x.
Claim: A contains an uncountable Δ-system with root $\emptyset$ — that is: an uncountable subcollection of pairwise disjoint elements.
Proof: Recursively choose, for $\xi < \omega_1$, sets $a_\xi \in A$ that are pairwise disjoint. This is possible: Suppose that, for some $\alpha < \omega_1$, pairwise disjoint a_ξ $(\xi < \alpha)$ have been found.
Claim: There exists $a \in A$ disjoint from $\bigcup_{\xi<\alpha} a_\xi$.
Proof: If not, this means that $\forall a \in A \exists x \in \bigcup_{\xi<\alpha} a_\xi (x \in a)$. Since A is uncountable and $\bigcup_{\xi<\alpha} a_\xi$ countable, (by a pigeon-hole principle) it follows that, for some $x \in \bigcup_{\xi<\alpha} a_\xi$, $\{a \in A \mid x \in a\}$ is uncountable, contradicting hypothesis. □

Corollary 8.47 *If* ZFC *is consistent, then so are* ZFC + ¬CH, $\text{ZFC} + 2^{\aleph_0} > \aleph_2$, $\text{ZFC} + 2^{\aleph_0} > \aleph_{\omega_1}, \ldots$

The rest of this section calculates the cardinality of $\wp(\omega)$ in generic extensions produced by the $\text{Fn}(\omega \times \lambda, 2)$. The following result also is a consequence of Lemma 8.33 (p. 89).

Lemma 8.48 *Every subset in* $\mathbf{M}[G]$ *of* $a \in \mathbf{M}$ *has a name* $\subset \{(b^\vee, p) \mid b \in a \wedge p \in P\}$ *in* $\mathbf{M}$.

Proof. Assume that $\pi_G \subset a$. Consider $\tau =_{\text{def}} \{(b^\vee, p) \mid b \in a \wedge p \Vdash b^\vee \in \pi\}$. Then $b \in \tau_G$ iff, (by definition of τ_G) $\exists p \in G(p \Vdash b^\vee \in \pi)$ iff, (by the Truth Lemma) $b \in \pi_G$. □

Thus, $\mathbf{M}[G] \cap 2^\omega$, the continuum of $\mathbf{M}[G]$, is the image (under the decoding map) of the set $\mathbf{M} \cap \wp(\{(n^\vee, p) \mid n \in \omega \wedge p \in P\})$ in $\mathbf{M}$. However, we can do better.

Definition 8.49 A name $\tau \in \mathbf{M}^P$ is a *nice name for a subset of* $a \in \mathbf{M}$ if $\tau = \bigcup_{b\in a}(\{b^\vee\} \times A_b)$, where every $A_b \subset P$ is an antichain.

NN(a) is the collection of nice names in $\mathbf{M}^P$ for subsets of a; note that $\text{NN}(a) \in \mathbf{M}$.

Thus, a nice name for a subset of a can be identified with a function $b \mapsto A_b$ on a in $\mathbf{M}$ that associates antichains with elements of a.

Lemma 8.50 *Every subset in* $\mathbf{M}[G]$ *of* $a \in \mathbf{M}$ *has a nice name.*

Proof. Suppose that $\pi_G \subset a$. Using Zorn's lemma in $\mathbf{M}$, choose, for every $b \in a$, a maximal antichain $A_b \subset \{p \mid p \Vdash b^\vee \in \pi\}$ and put $\tau =_{\text{def}} \bigcup_{b\in a}(\{b^\vee\} \times A_b)$.
$\tau_G \subset \pi_G$:

Suppose $b \in \tau_G$; say, $p \in A_b \cap G$. Then $b \in \pi_G$ follows by the Truth Lemma.
$\pi_G \subset \tau_G$:
Suppose $b \in \pi_G$; say, $p \in G$ forces $b^\vee \in \pi$. By the Truth Lemma, it suffices to show the
Claim: $p \Vdash b^\vee \in \tau$.
Proof: Instead, we check that $p \Vdash \neg\neg(b^\vee \in \tau)$, i.e.: $\forall q \leqslant p \exists r \leqslant q(r \Vdash b^\vee \in \tau)$. Thus, assume $q \leqslant p$. Extension Lemma: $q \Vdash b^\vee \in \pi$. By maximality of A_b, some $q' \in A_b$ is compatible with q; say, $r \leqslant q, q'$. By choice of A_b and the Completeness Lemma, $r \Vdash b^\vee \in \tau$. (For, if $G' \ni r$ is generic, then $q' \in G'$, $(b^\vee, q') \in \tau$, and $b \in \tau_{G'}$.) □

Corollary 8.51 *In* $\mathbf{M}[G]$, $\wp(a) \leqslant_1 \mathrm{NN}(a)$.

Corollary 8.52 *If (in* $\mathbf{M}$*)* $\lambda^\omega = \lambda$, *then* $\mathrm{Fn}(\omega \times \lambda, 2)$ *forces* $2^\omega = \lambda$ *(in* $\mathbf{M}[G]$*).*

Warning. Note that, in 8.52, we confuse the initial λ with its cardinal. In the sequel, we allow this confusion several times.

Proof. We saw before that, in $\mathbf{M}[G]$, $\lambda \leqslant 2^\omega$. As to the converse: by 8.51, it suffices to show that, in $\mathbf{M}$, $\mathrm{NN}(\omega) \leqslant_1 \lambda$. Note that $P =_1 \lambda$; and, since P is ccc, there are at most $\lambda^\omega = \lambda$ antichains in P and, hence, $\mathrm{NN}(\omega) \leqslant_1 \lambda^\omega = \lambda$. □

As to the conditon $\lambda^\omega = \lambda$, note:

Lemma 8.53 (GCH) $\mathrm{cf}(\lambda) > \omega \Rightarrow \lambda^\omega = \lambda$.

Proof. $\lambda^\omega = \bigcup_{\alpha<\lambda} \alpha^\omega \leqslant_1 \bigcup_{\alpha<\lambda} 2^{\alpha \times \omega} \leqslant_1 \lambda \times \lambda =_1 \lambda$. □

Theorem 8.54 *The following are examples of (relatively) consistent alternatives to CH:* $2^{\aleph_0} = \aleph_2$, $2^{\aleph_0} = \aleph_{12345}$, $2^{\aleph_0} = \aleph_{\omega_1}$, $2^{\aleph_0} = \aleph_{\omega_{\omega_1}}, \ldots$

However, $2^{\aleph_0} = \aleph_\omega$ is *not* consistent. (See Lemma 6.22.2 p. 45.) Even before Cohen, Gödel has stressed that independence of CH w.r.t. ZFC and the question whether CH is true are separate issues. According to [Dawson 98] he has, in the later years of his life, fruitlessly tried to argue that $2^{\aleph_0} = \aleph_2$ (or, at least, that $2^{\aleph_0} \geqslant \aleph_2$).

Exercises

237 ♣ Prove that forcing with $\mathrm{Fn}(\omega, 2)$ preserves GCH. (Thus, GCH + $\wp(\omega) \not\subset \mathbf{L}$ is relatively consistent.)

238 ♣ Suppose that, in $\mathbf{M}$, GCH holds and P is ccc. Show that, in corresponding generic extensions $\mathbf{M}[G]$, we have that $\forall \kappa \geqslant |P|(2^\kappa = \kappa^+)$.

239 ♣ Suppose that, in $\mathbf{M}$, GCH holds, and $P = \mathrm{Fn}(\omega \times \omega_\omega, 2)$. Compute the value of $2^{\aleph_0}$ in a generic extension $\mathbf{M}[G]$.

240 ♣ Suppose that P is ccc and $G \subset P$ is $\mathbf{M}$-generic. Let $C \subset \omega_1$ be a club in $\mathbf{M}[G]$. Show that a club $C' \subset \omega_1$ in $\mathbf{M}$ exists such that $C' \subset C$.
Hint. Pick $f \in \mathbf{M}[G]$ such that (*) $\forall \alpha < \omega_1(\alpha < f(\alpha) \in C)$. (Thus, f "witnesses unboundedness" of C.) By ccc-ness, $F \in \mathbf{M}$ exists such that $\forall \alpha(f(\alpha) \in F(\alpha) \leqslant_1 \aleph_0)$. ($F(\alpha)$ is a set of "possible values" of $f(\alpha)$. If $p \in G$ forces (*), define $F(\alpha) = \{\beta \mid \exists q \leqslant p(q \Vdash f(\alpha) = \beta)\}$. See the proof of Theorem 8.43.)

8.4 Faits divers

8.4.1 Forcing CH

Definition 8.55 $\mathrm{Fn}_\lambda(I,J) =_{\mathrm{def}} \{p \subset I \times J \mid p \text{ is a function and } |p| < \lambda\}$.

Thus, $\mathrm{Fn}(I,J) = \mathrm{Fn}_\omega(I,J)$.

The following proves consistency of CH by means of forcing; the proof requires looking at a new combinatorial property of posets.

Theorem 8.56 $\mathrm{Fn}_{\omega_1}(\omega_1, 2^\omega)$ *forces* CH *(over any ground model).*

Definition 8.57 P is *λ-closed* if, for every descending sequence $p_0 \geqslant p_1 \geqslant p_2 \geqslant \cdots \geqslant p_\xi \geqslant \cdots$ $(\xi < \alpha)$ of length $\alpha < \lambda$, there exists $p \in P$ that is $\leqslant p_\xi$ for all $\xi < \alpha$.

For instance, if λ is regular, then $\mathrm{Fn}_\lambda(I,J)$ is λ-closed: the union of the descending chain has power $< \lambda$ and, hence, is a condition as well.

Theorem 8.58 *Suppose that* $\mathbf{M} \models (\alpha < \lambda$ *and* P *is* λ*-closed). If* G *is* $\mathbf{M}$*-generic and* $B \in \mathbf{M}$*, then*

1. $\mathbf{M}[G] \cap B^\alpha = \mathbf{M} \cap B^\alpha$;

hence, P *preserves*

2. *regularity of initials* $\leqslant \lambda$*, and, hence:*
3. *all initials* $\leqslant \lambda$*,*
4. *powers of all initials* $< \lambda$*.*

Proof of Theorem 8.56.
$\mathrm{Fn}_{\omega_1}(\omega_1, 2^\omega)$ is ω_1-closed. Therefore, by Theorem 8.58, it preserves $\omega_1^{\mathbf{M}}$ and $\mathbf{M} \cap 2^\omega$. An $\mathbf{M}$-generic filter G gives rise to a surjection from $\omega_1^{\mathbf{M}[G]} = \omega_1^{\mathbf{M}}$ onto $(2^\omega)^{\mathbf{M}[G]} = \mathbf{M}[G] \cap 2^\omega = \mathbf{M} \cap 2^\omega$ in $\mathbf{M}[G]$. Thus, $\mathbf{M}[G] \models 2^\omega \leqslant \omega_1$. □

To prove Theorem 8.58, we need part 2 of the following exercise.

241 ♣ Exercise. If $B \in \mathbf{M}$, then

1. $p \Vdash \forall x \in B^\vee\, \Phi(x) \;\Leftrightarrow\; \forall b \in B (p \Vdash \Phi(b^\vee))$,
2. $p \Vdash \exists x \in B^\vee\, \Phi(x) \;\Rightarrow\; \exists q \leqslant p \exists b \in B (q \Vdash \Phi(b^\vee))$.

Proof of Theorem 8.58.
3. If the initial $\beta \leqslant \lambda$ is not preserved, that means that for some $\alpha < \beta$, a new surjection $f : \alpha \to \beta$ arises in the generic extension, which is excluded by 1.
2. Idem.
4. Similarly, if $\alpha < \lambda$, there can be no new function $f : \alpha \to 2$ in a generic extension.
1. Suppose that $\pi_G : \alpha \to B$ is a function in $(\mathbf{M}[G] \cap B^\alpha) - (\mathbf{M} \cap B^\alpha)$. By the Truth lemma, some $p \in G$ forces $(\pi : \alpha^\vee \to B^\vee) \wedge \pi \notin (B^\alpha)^\vee$. Recursively define, in $\mathbf{M}$, a descending sequence $p_0 \geqslant p_1 \geqslant p_2 \geqslant \cdots \geqslant p_\xi \geqslant \cdots$ of conditions together with a sequence $b_0, b_1, b_2, \ldots, b_\xi, \ldots$ of elements of B $(\xi < \alpha)$ such that $p_\xi \Vdash \pi(\xi^\vee) = (b_\xi)^\vee$. Indeed, this

is possible: if p_ξ, b_ξ have been found for $\xi < \beta$ (where $\beta < \alpha$), we first pick $q \leqslant$ all p_ξ. Then $q \Vdash \exists x \in B^\vee(\pi(\beta^\vee) = x)$, and hence, by Exercise 241.2, we find $p_\beta \leqslant q$ and $b \in B$ such that $p_\beta \Vdash \pi(\beta^\vee) = b^\vee$ as required. The sequences constructed, choose $q \leqslant$ all p_ξ and define $f : \alpha \to B$ by $f(\xi) = b_\xi$. Then $q \Vdash \pi = f^\vee \wedge f^\vee \in (B^\alpha)^\vee$ (look what happens in generic extensions $\mathbf{M}[G']$ with $q \in G'$) — a contradiction. □

8.4.2 Two relations between posets

We have a look at two relations between posets and their effect on the resulting generic extensions.

242 ♣ Exercise. Suppose that $h : P \to Q$ is an isomorphism in $\mathbf{M}$ between the posets P and Q. Then

1. h induces a 1–1 correspondence between the dense sets in P and those in Q,
2. h induces a 1–1 correspondence between the $\mathbf{M}$-generic filters $\subset P$ and those $\subset Q$,
3. P and Q produce the same generic extensions,
4. h induces a 1–1 correspondence $h^* : \mathbf{M}^P \to \mathbf{M}^Q$ via the recursion $h^*(\pi) = \{(h^*(\tau), h(p)) \mid (\tau, p) \in \pi\}$,
5. $p \Vdash_P \Phi(\pi_1, \dots, \pi_k) \Leftrightarrow h(p) \Vdash_Q \Phi(h^*(\pi), \dots, h^*(\pi_k))$.

Thus, P-forcing and Q-forcing amount to the same thing.

Example: $\omega \times \omega_2 =_1 \omega_2$, thus: $\mathrm{Fn}(\omega \times \omega_2, 2) \cong \mathrm{Fn}(\omega_2, 2)$. (The first poset "adds ω_2 subsets of ω, the second one "adds one subset of ω_2".)

Lemma 8.59 *Suppose that the poset $P \subset Q$ is dense in the poset Q. Then:*

1. *if $D \subset P$ is dense in P, then D is dense in Q,*
2. *if $D \subset Q$ is dense in Q, then $\{p \in P \mid \exists q \in D(q \leqslant p)\}$ is dense in P,*
3. *if $G \subset Q$ is $\mathbf{M}$-generic over Q, then $G\Downarrow =_{def} G \cap P$ is $\mathbf{M}$-generic over P and, hence, $\mathbf{M}[G] = \mathbf{M}[G\Downarrow]$,*
4. *if $G \subset P$ is $\mathbf{M}$-generic over P, then $G\Uparrow =_{def} \{q \in Q \mid \exists p \in G(p \leqslant q)\}$ $\mathbf{M}$-generic over Q and, hence again, $\mathbf{M}[G] = \mathbf{M}[G\Uparrow]$,*
5. *the operations $\downarrow$ and $\uparrow$ are inverses of one another.*

243 ♣ Exercise. Relate P-forcing to Q-forcing when P is a dense part of Q.

Example 8.60 $\mathrm{Fn}_{\omega_1}(\omega_1, 2^\omega)$ (the poset from Theorem 8.56) is isomorphic with a dense part of $\mathrm{Fn}_{\omega_1}(\omega \times \omega_1, 2) \cong \mathrm{Fn}_{\omega_1}(\omega_1, 2)$.

Example 8.61 Suppose that B is *the* countable, non-atomic boolean algebra. Let P be the set of non-0-elements of B with the inherited ordering. Every countable poset Q that satisfies $\forall q \exists r, r' \leqslant q(r \perp r')$ is isomorphic with a dense part of P. Thus, the posets $\mathrm{Fn}(\omega, 2)$, $\mathrm{Fn}(\omega, \omega)$, $\bigcup_n 3^n$ etc. all produce the same generic extensions.

8.4.3 A few conventions

Suppose that P is a poset (in $\mathbf{M}$).
1. For any $a \in \mathbf{M}$, $a^\vee \in \mathbf{M}^P$ is a standard name denoting a in any forcing extension of $\mathbf{M}$. We can agree to use, in the future, the elements of $\mathbf{M}$ themselves as names. This could be ambiguous since a name $\pi \in \mathbf{M}^P \subset \mathbf{M}$ will denote something different from π, but the context will always indicate what is meant. This would save us writing $^\vee$ dozens of times.

A nice name for a subset of $a \in \mathbf{M}$ would then be written $\bigcup_{b \in a}(\{b\} \times A_b)$.
2. Elements of generic extensions have many different names. We can agree in future that $\mathbf{a}$ is some name of $a \in \mathbf{M}[G]$. This would save a lot of subscripts $_G$.
3. As will be clear by now, forcing is a method that reduces questions about $\mathbf{M}[G]$ to questions about (the poset and) $\mathbf{M}$. Thus, $\mathbf{M}$ is the place where 99% of the arguments "take place". It is tiresome to constantly have to say "now, in $\mathbf{M}$,..." and phrases that are similar. Therefore, authors often pretend that $\mathbf{M} = \mathbf{V}$ and behave as if they were constructing forcing extensions $\mathbf{V}[G]$ of the universe $\mathbf{V}$ instead of some countable transitive model $\mathbf{M}$. Clearly, there is nothing against this as–if attitude as long as you realize what is going on. This saves a lot of relativizing-to-$\mathbf{M}$.

8.4.4 Forcing ¬AC

At first sight, this looks problematic since AC is preserved by forcing. However, we can force $\mathbf{M}[G]$ to have a *submodel* in which AC is false. For this, we have to generalize consructibility somewhat.

Definition 8.62 $\mathbf{L}(A) = \bigcup_\alpha \mathrm{L}_\alpha(A)$, where the $\mathrm{L}_\alpha(A)$ are defined in the same way as the L_α, except that $\mathrm{L}_0(A) = A$.

Lemma 8.63 *If A is transitive, then so is $\mathbf{L}(A)$, and $\mathbf{L}(A) \models$ ZF.*

Proof. Exactly as in the case of $\mathbf{L}$. □

The submodel that falsifies AC is $\mathbf{L}(\wp\omega)$ (in a suitable generic extension).

Definition 8.64 A poset P is *almost homogeneous* if, for all $p, q \in P$, there exists an order-automomorphism h of P such that $p \sim h(q)$.

All posets that we came across up to now are almost homogeneous. E.g., if I is infinite, then $\mathrm{Fn}(I, 2)$ is an example: suppose $p, q \in \mathrm{Fn}(I, 2)$. Choose a permutation j of I such that $j[\mathrm{Dom}(q)] \cap \mathrm{Dom}(p) = \emptyset$. j induces an automorphism h of $\mathrm{Fn}(I, 2)$. By choice of j, $\mathrm{Dom}(p) \cap \mathrm{Dom}(h(q)) = \emptyset$. Thus, $p \sim h(q)$.

Lemma 8.65 *Suppose that P is almost homogenous, $p \in P$, $\Phi = \Phi(x)$ and $a \in \mathbf{M}$. Then $p \Vdash \Phi(a^\vee) \Leftrightarrow 1 \Vdash \Phi(a^\vee)$.*

Proof. $\Leftarrow$: By the Extension lemma. $\Rightarrow$: Assume that $p \Vdash \Phi(a^\vee)$ but $1 \not\Vdash \Phi(a^\vee)$. Choose $q \leqslant 1$ s.t. $q \Vdash \neg\Phi(a^\vee)$. Let $r \leqslant p, h(q)$, where h is an automorphism of P. Note, that $h^*(a^\vee) = a^\vee$ (names $a^\vee$ only use the condition $1 \in P$ and $h(1) = 1$). Thus, $h(q) \Vdash \neg\Phi(a^\vee)$. Thus, r forces both $\Phi(a^\vee)$ and $\neg\Phi(a^\vee)$, a contradiction. □

Lemma 8.66 *Suppose that I and J are uncountable, $P = \mathrm{Fn}(I, 2)$, $Q = \mathrm{Fn}(J, 2)$, $\alpha \in$ OR. Then $1 \Vdash_P [\Phi(\alpha^\vee)]^{\mathbf{L}(\wp\omega)}$ iff $1 \Vdash_Q [\Phi(\alpha^\vee)]^{\mathbf{L}(\wp\omega)}$.*

Proof. If, by coincidence, $|I| = |J|$, then $P \cong Q$, and the result is trivial. Thus, suppose that $|I| < |J|$. First, we force $|I| = |J|$ using the poset $R = \mathrm{Fn}_{\omega_1}(I, J)$. Let $H \subset R$ be $\mathbf{M}$-generic. Then $|I| = |J|$ in $\mathbf{M}[H]$. Now if $G \subset P$ is $\mathbf{M}[H]$-generic, then G is a fortiori $\mathbf{M}$-generic, and we have the

Claim 1. $\mathbf{M}[H][G] \cap \wp\omega = \mathbf{M}[G] \cap \wp\omega$.
Proof. $\supset$: By minimality of $\mathbf{M}[G]$, $\mathbf{M}[G] \subset \mathbf{M}[H][G]$. $\subset$: Let $\pi \in \mathbf{M}[H]^P$ be a nice name for a subset of ω in $\mathbf{M}[H][G]$. Say, $\pi = \bigcup_n(\{n^\vee\} \times A_n)$ where the $A_n \subset P$ are antichains. Since P is ccc (in any model), the A_n are countable in $\mathbf{M}[H]$. Since R is ω_1-closed relative $\mathbf{M}$, all A_n are in $\mathbf{M}$. For the same reason, $\pi \in \mathbf{M}^P$. Thus, $\pi_G \in \mathbf{M}[G]$.

Claim 2. $1 \Vdash_{P,\mathbf{M}} [\Phi(\alpha^\vee)]^{\mathbf{L}(\wp\omega)}$ iff $1 \Vdash_{P,\mathbf{M}[H]} [\Phi(\alpha^\vee)]^{\mathbf{L}(\wp\omega)}$.

This implies the required equivalence. For, a similar Claim is true w.r.t. Q; however, we have that, since $P \cong Q$ in $\mathbf{M}[H]$: $1 \Vdash_{P,\mathbf{M}[H]} [\Phi(\alpha^\vee)]^{\mathbf{L}(\wp\omega)}$ iff $1 \Vdash_{Q,\mathbf{M}[H]} [\Phi(\alpha^\vee)]^{\mathbf{L}(\wp\omega)}$.

Proof of Claim 2: $\Rightarrow$: Suppose that $G \subset P$ is $\mathbf{M}[H]$-generic. Then G is also $\mathbf{M}$-generic, and by hypothesis $[\Phi(\alpha)]^{\mathbf{L}(\wp\omega)}$ is true in $\mathbf{M}[G]$. By Claim 1, this statement is also true in $\mathbf{M}[H][G]$. $\Leftarrow$: Otherwise, choose $p \in P$ s.t. $p \Vdash_{P,\mathbf{M}} \neg[\Phi(\alpha^\vee)]^{\mathbf{L}(\wp\omega)}$. Choose a generic $G \ni p$, etc. □

Theorem 8.67 *If I is uncountable, then* $\mathrm{Fn}(I, 2)$ *forces that $\wp\omega$ doesn't have a wellordering in $\mathbf{L}(\wp\omega)$.*

Proof. Otherwise, choose a condition p that forces $[\wp\omega =_1 \kappa^\vee]^{\mathbf{L}(\wp\omega)}$, where $\alpha \in \mathrm{OR}$. Since $P = \mathrm{Fn}(I, 2)$ is almost homogeneous, this is forced by 1 as well. Choose a set $J >_1 \alpha$ and consider $Q = \mathrm{Fn}(J, 2)$. By Lemma 8.66, the statement is also Q-forced by $1 \in Q$. Let $G' \subset Q$ be $\mathbf{M}$-generic. Then in $\mathbf{M}[G']$ are true: $[\wp\omega =_1 \alpha^\vee]^{\mathbf{L}(\wp\omega)}$, and, hence (i) $\wp\omega =_1 \alpha^\vee$; (ii) (since Q is ccc) $\alpha <_1 J$; (iii) (since $Q \cong \mathrm{Fn}(\omega \times J, 2)$) $\wp\omega \geqslant_1 J$. But (i), (ii) and (iii) together are contradictory. □

8.4.5 Alternatives to GCH

We can use the $\mathrm{Fn}_\lambda(\kappa, 2)$ to blow up 2^λ and simultaneously preserve powers 2^μ for $\mu < \kappa$ in much the same way as $\mathrm{Fn}(\kappa, 2)$ blows up 2^ω. (If we don't mind preserving powers of smaller cardinals, then $\mathrm{Fn}(\kappa, 2)$ is fine as well.)

Theorem 8.68 *If λ is regular, $\forall\mu < \lambda(2^\mu \leqslant \lambda)$, $\lambda < \kappa$, and $\kappa^\lambda = \kappa$, then* $\mathrm{Fn}_\lambda(\kappa, 2)$*:*

1. *preserves regularity,*
2. *preserves all powers 2^μ where $\mu < \lambda$,*
3. *forces $2^\lambda = \kappa$.*

Example 8.69 ZFC is consistent with $2^\omega = \omega_1$ (CH) and $2^{\omega_1} = \omega_3$.
Proof. Start with $\mathbf{M}$ satisfying GCH. Note that by GCH, $\omega_3^{\omega_1} = (2^{\omega_2})^{\omega_1} = 2^{\omega_2} = \omega_3$. Use $\mathrm{Fn}_{\omega_1}(\omega_3, 2)$; this preserves CH and blows up 2^{ω_1} to ω_3.

Definition 8.70 A poset is λ-*cc* if it has no antichain of power λ.

Thus, ccc equals ω_1-cc.

Proof of Theorem 8.68.
3a. $P = \mathrm{Fn}_\lambda(\kappa, 2) \cong \mathrm{Fn}_\lambda(\kappa \times \lambda, 2)$, thus, as in Exercise 207 (p. 81), P forces $2^\lambda \geqslant \kappa$.
1a. Since λ is regular, the poset is λ-closed; thus regularity of initials $\leqslant \lambda$ is preserved.
2. For the same reason, powers 2^μ $(\mu < \lambda)$ are preserved.
1b. By Corollary 8.72, P is λ^+-cc. Thus, by an argument that generalizes Theorem 8.43 (p. 93): regularity of initials $> \lambda$ is preserved.
3b. $|P| \leqslant \kappa^\lambda = \kappa$. Since P is λ^+-cc, there are at most $\kappa^\lambda = \kappa$ antichains and hence at most $\kappa^\lambda = \kappa$ nice names for subsets of λ. Thus, $2^\lambda \leqslant \kappa$ is forced. □

244 ♣ Exercise. Generalize Theorem 8.43 (p. 93) to: if P is λ^+-cc, it preserves regularity of initials $> \lambda$.

Note: If λ is regular, and $\forall \mu < \lambda (2^\mu \leqslant \lambda)$, then $\forall \mu < \lambda (\lambda^\mu = \lambda)$.
For: $\lambda^\mu = \sum_{\alpha<\lambda} \alpha^\mu \leqslant \sum_{\alpha<\lambda} 2^{\alpha\mu} \leqslant \sum_{\alpha<\lambda} \lambda = \lambda$.
[Kunen 80] has refinements for the following two results.

Lemma 8.71 (Δ-systems lemma 2) *If* $\forall \mu < \lambda (2^\mu \leqslant \lambda)$*, then every collection of* λ^+ *sets of power* $< \lambda$ *includes a* Δ*-system of power* λ^+*.*

Corollary 8.72 *If* $\forall \mu < \lambda (2^\mu \leqslant \lambda)$*, then* $\mathrm{Fn}_\lambda(\kappa, 2)$ *is* λ^+*-cc.*

Proof. Suppose that $A \subset \mathrm{Fn}_\lambda(\kappa, 2)$ is an antichain of power λ^+. Distinguish two cases. (i) $\{\mathrm{Dom}(p) \mid p \in A\} \leqslant \lambda$. By a pigeon-hole principle, for some w, we have that $\{p \in A \mid \mathrm{Dom}(p) = w\} > \lambda$. But, $w <_1 \lambda$. Thus, $2^w \leqslant_1 \lambda$, a contradiction. (ii) $\{\mathrm{Dom}(p) \mid p \in A\} > \lambda$. Apply Lemma 8.71. □

Proof of Lemma 8.71.
Suppose that A is a collection of power λ^+ of sets of power $< \lambda$. Then $|\bigcup A| \leqslant \sum_{a\in A} |a| \leqslant \sum_{a\in A} \lambda = \lambda^+$, so we may assume w.l.o.g. that $\bigcup A \subset \lambda^+$.
We can't have that $|\bigcup A| \leqslant \lambda$, since $|\{a \subset \lambda \mid |a| < \lambda\}| \leqslant \sum_{\mu<\lambda} \lambda^\mu = \sum_{\mu<\lambda} \lambda = \lambda$. Thus, w.l.o.g., we may assume that $\bigcup A = \lambda^+$.
Consider the map : $A \to \lambda$ that associates with every $a \in A$ its order type. By a pigeon-hole principle, we may w.l.o.g. assume that every $a \in A$ has the same type $\alpha < \lambda$.
Let ξ_a be the ξ-th element of $a \in A$. Define $A_\xi = \{\xi_a \mid a \in A\}$. So, $\lambda^+ = \bigcup A = \bigcup_{\xi<\alpha} A_\xi$, and, since λ^+ is regular, there is a least ordinal $\xi^0 < \alpha$ such that A_{ξ^0} is cofinal in λ^+.
But then, $\bigcup_{\xi<\xi^0} A_\xi$ is bounded in λ^+, say, by α_0.
Consider the map $a \mapsto d{\downarrow} =_{\mathrm{def}} \{\xi_a \mid \xi < \xi^0\}$ defined on A. $d{\downarrow}$ is a subset of α_0 of type ξ^0, and $\alpha_0^{\xi^0} \leqslant_1 \lambda^{\xi^0} =_1 \lambda$. By a pigeon-hole principle, for some $d \subset \alpha_0$ of type ξ^0 we have that $A^* =_{\mathrm{def}} \{a \in A \mid d{\downarrow} = d\} =_1 \lambda^+$.
In A^* we construct a sequence $\{a_\delta\}_{\delta<\lambda^+}$ such that $\delta < \delta' \Rightarrow \bigcup a_\delta < \xi^0_{a_{\delta'}}$. This is possible by choice of ξ^0. This is the required Δ-system with root d. □

245 ♣ Exercise. ZFC is consistent with $2^\omega = \omega_1$, $2^{\omega_1} = \omega_3$ and $2^{\omega_2} = \omega_7$ simultaneously.
Proof. Assume GCH in the model you start from. The trick is to work backwards. (If you start with $\mathrm{Fn}_{\omega_1}(\omega_3, 2)$, you force $2^\omega = \omega_1$, $2^{\omega_1} = 2^{\omega_2} = \omega_3$. But, in order for $\mathrm{Fn}_{\omega_2}(\omega_7, 2)$ to work, you need $2^{\omega_1} \leqslant \omega_2$.) Thus, *first* force with $\mathrm{Fn}_{\omega_2}(\omega_7, 2)$. *Next*, force with $\mathrm{Fn}_{\omega_1}(\omega_3, 2)$.

Note that, if λ is regular and $< \mathrm{cf}(\kappa)$, the necessary conditions $\forall\mu < \lambda(2^\mu \leqslant \lambda)$ and $\kappa^\lambda = \kappa$ from Theorem 8.68 follow from GCH. For the first one, this is obvious. For the second one, $\kappa^\lambda \leqslant \sum_{\mu<\kappa} \mu^\lambda \leqslant \sum_{\mu<\kappa} 2^{\mu\lambda} \leqslant \sum_{\mu<\kappa} \kappa = \kappa$. Thus, the answer forcing provides for the possible values of 2^λ for *regular* λ fully complements the ZFC-result that $\mathrm{cf}(2^\lambda) > \lambda$.

The strongest result, for all regular cardinals simultaneously, is Easton's (1964): if you have a definable operation f, mapping regular cardinals to cardinals, such that, for regular λ, $\mathrm{cf}(f(\lambda)) > \lambda$ (and of course, such that $\lambda < \lambda \Rightarrow f(\lambda) \leqslant f(\lambda'))$, then you can force $2^\lambda = f(\lambda)$ for all regular λ simultaneously and preserve cardinals. (By Exercise 238 p. 95, this cannot be forced with just one poset.)

The *singular cardinals problem* is the question whether we can, similarly, change freely the powers of singular cardinals. A few answers are Exercise 119 (p. 46) and Silver's Theorem 6.36 (p. 50; for a proof via exercises, see [Kunen 80]).

Exercise 119 computes 2^λ when $\mathrm{cf}(\lambda) < \lambda$ and the map $\mu \mapsto 2^\mu$ becomes stationary below λ. If it doesn't become stationary, then $\mathrm{cf}(\sum_{\mu<\lambda} 2^\mu) = \mathrm{cf}(\lambda)$, and it follows (since $\mathrm{cf}(2^\lambda) > \lambda$) that $2^\lambda > \sum_{\mu<\lambda} 2^\mu$. Jensen showed it requires a *measurable cardinal* to prove consistency of $2^\lambda \neq (\sum_{\mu<\lambda} 2^\mu)^+$, and even stronger results are known.

8.5 Iterated forcing

8.5.1 Products

The purpose here is to show: if $G_1 \subset P_1 \in \mathbf{M}$ is $\mathbf{M}$-generic, and $G_2 \subset P_2 \in \mathbf{M}[G_1]$ is $\mathbf{M}[G_1]$-generic, then a $P_3 \in \mathbf{M}$ and an $\mathbf{M}$-generic $G_3 \subset P_3$ exist such that $\mathbf{M}[G_1][G_2] = \mathbf{M}[G_3]$: forcing twice equals forcing once.

To make things easier, we'll first assume that $P_2 \in \mathbf{M}$. (The case $P_2 \in \mathbf{M}[G_1]$ is treated in Theorem 8.90.)

Definition 8.73 (*The product-order.*) For posets P and Q, define $\leqslant$ on $P \times Q$ by

$$(p, q) \leqslant (p', q') \equiv_{\mathrm{def}} p \leqslant_P p' \ \wedge\ q \leqslant_Q q'.$$

Theorem 8.74 (Product lemma, simple version) *Suppose that P and Q are posets in $\mathbf{M}$. $P \times Q$-Forcing is the same as P-forcing followed by Q-forcing. More precisely:*

1. *if $F \subset P$ is $\mathbf{M}$-generic and $G \subset Q$ is $\mathbf{M}[F]$-generic, then $F \times G \subset P \times Q$ is $\mathbf{M}$-generic and $\mathbf{M}[F \times G] = \mathbf{M}[F][G]$;*

2. *if $H \subset P \times Q$ is $\mathbf{M}$-generic, then $F =_{def} \mathrm{Dom}(H) \subset P$ is $\mathbf{M}$-generic, $G =_{def} \mathrm{Ran}(H) \subset Q$ is $\mathbf{M}[F]$-generic, and $\mathbf{M}[H] = \mathbf{M}[F][G]$.*

Note that, by symmetry, in 8.74.2, F will be $\mathbf{M}[G]$-generic as well.

Proof. Part 2. First, check that $F = \{p \mid (p, 1) \in H\}$ and $G = \{q \mid (1, q) \in H\}$. Thus, $H = F \times G$. To see that G is $\mathbf{M}[F]$-generic, choose $D \in \mathbf{M}[F]$ dense. Let $p' \in F$ force $\mathbf{D}$ is dense.

Claim. $\{(p, q) \mid p \leqslant p' \ \wedge\ p \Vdash q \in \mathbf{D}\}$ is dense below $(p', 1)$.

Proof. Assume $(p_1, q_1) \leqslant (p', 1)$. Thus, p_1 forces $\mathbf{D}$ is dense, i.e., p_1 forces $\exists x(x \in \mathbf{D} \wedge x \leqslant q_1)$. Thus, for some $p \leqslant p_1$ and $q \in Q$, $p \Vdash (q \in \mathbf{D} \ \wedge\ q \leqslant q_1)$.

As a result, since $(p', 1) \in H$, H intersects $\{(p, q) \mid p \leqslant p' \ \wedge\ p \Vdash q \in \mathbf{D}\}$, say, in (p, q). Thus, $p \in F$, $q \in G$, $p \Vdash q \in \mathbf{D}$, and $q \in D \cap G$. □

246 ♣ Exercise. Show that, in 8.74.1, it doesn't suffice to require that G is $\mathbf{M}$-generic.

247 ♣ Exercise. Prove 8.74.1.
Hint. Suppose that $D \subset P \times Q$ is dense in $\mathbf{M}$. Note: in order that $\exists p \in F \exists q \in G((p,q) \in D)$, it suffices that $\{q \mid \exists p \in F((p,q) \in D)\}$ is dense.

248 ♣ Exercise. Show that $\forall n(2^{\omega_n} = \omega_{n+2})$ is consistent.
Hint. The previous section doesn't apply here. (How force infinitely many times? How work backwards?) Start with $\mathbf{M}$ satisfying GCH. Use $\prod_n \mathrm{Fn}_{\omega_n}(\omega_{n+2}, 2)$. To see that $2^{\omega_m} = \omega_{m+2}$ is forced, consider the factorization $\prod_n \mathrm{Fn}_{\omega_n}(\omega_{n+2}, 2) \cong \prod_{n \leqslant m} Fn_{\omega_n}(\omega_{n+2}, 2) \times \prod_{m<n} Fn_{\omega_n}(\omega_{n+2}, 2)$ and use the Product lemma.

Note that, if $I = I_1 \cup I_2$ is a partition if I, then $\mathrm{Fn}(I,2) \cong \mathrm{Fn}(I_1,2) \times \mathrm{Fn}(I_2,2)$; thus the Product lemma applies to $\mathrm{Fn}(I,2)$-forcing. Furthermore, if $H \subset \mathrm{Fn}(I_1,2) \times \mathrm{Fn}(I_2,2)$, then $\mathrm{Dom}(H) = H \cap \mathrm{Fn}(I_1,2)$ and then $\mathrm{Ran}(H) = H \cap \mathrm{Fn}(I_2,2)$.

Lemma 8.75 *If $H \subset \mathrm{Fn}(I,2)$ is $\mathbf{M}$-generic and $a \in \mathbf{M}[H] \cap \wp\omega$, then a $\mathbf{M}$-countable $J \subset I$ exists such that $a \in \mathbf{M}[H \cap \mathrm{Fn}(J,2)]$.*

Proof. Let $\tau = \bigcup_n(\{n\} \times A_n)$ be a nice $\mathrm{Fn}(I,2)$-name for a. By ccc, all A_n are countable. Thus, $A = \bigcup_n A_n$ is countable. Thus $J =_{\mathrm{def}} \bigcup\{\mathrm{Dom}(p) \mid p \in A\}$ is countable, τ is a $\mathrm{Fn}(J,2)$-name, and $a \in \mathbf{M}[H \cap \mathrm{Fn}(J,2)]$. □

The lemma can be used to obtain a consistency result concerning the cardinality of *almost disjoint families*. A family $A \subset \wp\omega$ is *almost disjoint* (*a.d.*) if the intersection of any two different elements of A is finite.

Note: a pairwise *disjoint* collection $\subset \wp\omega$ is necessarily countable, but an a.d. family of power 2^ω is easy to construct: replace ω by $\mathbb{Q}$ and, for every irrational, choose an increasing sequence of rationals that converges to it. Two different sequences must diverge eventually.

Lemma 8.76 *Every m.a.d. (= maximal a.d.) family is uncountable.*

Proof. Suppose that $\{a_n \mid n \in \omega\}$ is an a.d. family of infinite sets $\subset \omega$. (Finite sets can be disregarded since there are only countably many.) Then the sets $b_n = a_n - \bigcup_{m<n} a_m$ are non-empty and pairwise disjoint. Choose $i_n \in b_n$ and consider $a =_{\mathrm{def}} \{i_n \mid i \in \omega\}$. We have that $m < n \Rightarrow i_n \notin a_m$ and, hence, $i_n \in a_m \Rightarrow n \leqslant m$; i.e., $a \cap a_m \subset \{i_0, \ldots, i_m\}$, and $\{a_n \mid n \in \omega\}$ is not m.a.d. □

Theorem 8.77 $\neg$CH *is consistent with the existence of a m.a.d. family of power ω_1.*

Proof. Start with $\mathbf{M}$ satisfying CH. In $\mathbf{M}$, we produce a m.a.d. family $\mathcal{A} = \{A_\xi \mid \xi < \omega_1\}$ that stays m.a.d. in generic extensions induced by posets $\mathrm{Fn}(I,2)$. For $|I| = \omega_2$, the result follows.

If $\mathcal{A}$ isn't m.a.d. any longer in $\mathbf{M}[H]$ (where $H \subset \mathrm{Fn}(I,2)$ is generic), some $a \in \mathbf{M}[H] \cap \wp\omega$ is a.d. with every A_ξ. By Lemma 8.75, for some countable $J \subset I$, $a \in \mathbf{M}[H \cap \mathrm{Fn}(J,2)]$. But, $\mathrm{Fn}(J,2) \cong \mathrm{Fn}(\omega,2)$; so we need to check maximality of $\mathcal{A}$ only in generic extensions via $P = \mathrm{Fn}(\omega,2)$.

Construction of $\mathcal{A}$: In $\mathbf{M}$, fix an enumeration $\{(p_\xi, \tau_\xi) \mid \xi < \omega_1\}$ of all pairs (p,τ), where $p \in P$ and τ is a nice name for a subset of ω. (Use CH.) Choose the first ω A_n pairwise disjoint. Suppose $\omega \leqslant \xi < \omega_1$ and A_σ constructed for $\sigma < \xi$. Choose A_ξ such that

1. A_ξ is a.d. with all A_σ $(\sigma < \xi)$,

2. *if* (i) $p_\xi \Vdash \tau_\xi =_1 \omega$ and $\forall\sigma < \xi(p_\xi \Vdash [\tau_\xi \cap A_\sigma <_1 \omega])$,
   then (ii) $\forall n \forall q \leqslant p_\xi \exists r \leqslant q \exists m \geqslant n(m \in A_\xi \wedge r \Vdash m \in \tau_\xi)$.

If this succeeds, then $\mathcal{A} = \{A_\xi \mid \xi < \omega_1\}$ is as required: if it happens to be not m.a.d. any longer in some extension, then for some ξ we have $p_\xi \Vdash [(\tau_\xi \geqslant_1 \omega \wedge \forall X \in \mathcal{A}(\tau_\xi \cap X <_1 \omega)]$. A fortiori, 2(i) holds, and $p_\xi \Vdash (\tau_\xi \cap A_\xi <_1 \omega)$. Thus, $p_\xi \Vdash \exists n(\tau_\xi \cap A_\xi \subset n)$. So, for some $q \leqslant p_\xi$ and n we have that $q \Vdash \tau_\xi \cap A_\xi \subset n$, contradicting 2(ii).
This succeeds: if (i) fails, it is no problem finding A_ξ. So suppose (i) is true. Let $\{B_i \mid i \in \omega\}$ be a re-numbering of $\{A_\sigma \mid \sigma < \xi\}$ and $i \mapsto (n_i, q_i)$ an enumeration of $\omega \times \{q \mid q \leqslant p_\xi\}$. By (i), $q_i \Vdash [\tau_\xi - (B_0 \cup \cdots \cup B_i) \geqslant_1 \omega]$. Choose $r_i \leqslant q_i$ and $m_i \geqslant n_i$ s.t. $m_i \notin B_0 \cup \cdots \cup B_i$ and $r_i \Vdash m_i \in \tau_\xi$. Take $A_\xi =_{\text{def}} \{m_i \mid i \in \omega\}$. (Compare 8.76.) $\square$

8.5.2 Martin's axiom

Suslin's hypothesis. Cantor showed that $(\mathbb{R}, <)$ is, up to isomorphism, the only linear ordering without endpoints that is separable (there is a countable subset that intersects every non-empty open interval) and complete (every non-empty set with an upper bound has a *least* upper bound).

Separability implies that every collection of pairwise disjoint open intervals must be countable. In 1920, Suslin asked whether Cantor's theorem remains true if one replaces separability by the latter condition. Suslin's hypothesis (SH) states that the answer is positive.

A *Suslin line* is a counter-example to SH, that is: a complete linear ordering in which every collection of pairwise disjoint open intervals is countable, but which is not separable. In 1967, assuming $\mathbf{V} = \mathbf{L}$, Jensen constructed a Suslin line. In fact, he showed that (i) $\mathbf{V} = \mathbf{L} \Rightarrow \Diamond$ and (ii) $\Diamond \Rightarrow \neg$SH, where $\Diamond$ ("diamond") states that a $\Diamond$-*sequence* $\{a_\xi \mid \xi < \omega_1\}$ exists, i.e., $a_\xi \subset \xi$, and for every $a \subset \omega_1$, $\{\xi \mid \xi \cap a = a_\xi\}$ is *stationary*, that is: intersects every club subset of ω_1.

Since stationary implies unbounded, every $a \subset \omega$ must occur in a $\Diamond$-sequence; and hence, $\Diamond \Rightarrow$ CH. The same poset $\mathrm{Fn}_{\omega_1}(\omega_1, 2)$ that forces CH happens to force $\Diamond$ as well.

In 1970, Solovay and Tennenbaum initiated "iterated forcing" in order to prove consistency of the Suslin hypothesis. The idea is as follows. Given a Suslin line, it is possible to transform it in a *Suslin tree*, that is: a tree of height ω_1 in which all chains and antichains are countable. It is easy to "kill" such a tree by forcing, generically adding a chain of length ω_1: if $(T, \geqslant)$ is a tree of height ω_1, a generic filter G will intersect all dense sets $D_\xi = \{p \in T \mid$ the height of p in T is $\geqslant \xi\}$ $(\xi < \omega_1)$. But, such a filter is nothing but a branch of length ω_1. However, it doesn't suffice to kill, generically, all Suslin trees in the ground model, since new Suslin trees may arise in the extension. And this clarifies the need to iterate.

Martin noticed that the Solovay-Tennenbaum argument also proved consistency of a new principle, which later got the name *Martin's axiom*.

Definition 8.78 (*Martin's axiom.*)

1. $\mathrm{MA}(\kappa) \equiv_{\text{def}}$ whenever P is ccc and $\mathcal{D}$ is a collection of $\leqslant \kappa$ sets dense in P, there exists a filter that intersects all of them.

2. $\mathrm{MA} \equiv_{\text{def}} \forall\kappa < 2^\omega(\mathrm{MA}(\kappa))$.

Note that MA plus ¬CH forbids Suslin trees by the argument given above.

MA has equivalents in terms of boolean algebras and of Hausdorff spaces. It is popular in set-theoretic topology. There is a fat monograph of Fremlin on MA that deals with its consequences.

Applications of MA are reminiscent of forcing. Intuitively, MA states that the universe is closed under forcing extensions in a weak sense: in an extension, there is always a filter (whether the poset is ccc or not) that intersects all dense sets in the ground universe; MA gives you a filter *in the universe itself* that intersects not too many dense sets, provided the poset is ccc. Therefore, MA is also called an *internal forcing axiom.* Newer generations of such axioms are the Proper forcing axiom and Martin's maximum.

Remark 8.79 *We have the following:*

1. MA(ω),
2. *CH* $\Rightarrow$ MA,
3. MA(κ) $\Rightarrow$ $\kappa < 2^\omega$,
4. *without cc-condition,* MA(ω_1) *is contradictory,*
5. MA $\Rightarrow$ *every m.a.d. family is* $=_1 2^\omega$,
6. MA(κ) $\Rightarrow$ $2^\kappa = 2^\omega$,
7. MA $\Rightarrow$ 2^ω *is regular (but, in a sense, that is all that is implied about* 2^ω*)*

Proof. For parts 5, 6 and 7, see [Kunen 80]. For the rest, see Exercise 249. □

249 ♣ Exercise Prove 8.79.1, 2, 3 and 4.
Hints. 1 does not need ccc. For 2, use Fn(ω, 2). For 4, use Fn(ω, ω_1).

Lemma 8.80 *Restricting* MA(λ) *to posets of power* $\leqslant \lambda$ *doesn't weaken it.*

Proof. Suppose that $\mathcal{D}$ is a collection of λ dense sets in the ccc poset $(P, \leqslant)$. Consider the *model* $(P, \leqslant, D)_{D \in \mathcal{D}}$. By Löwenheim-Skolem, $(P', \leqslant, P' \cap D)_{D \in \mathcal{D}} \prec (P, \leqslant, D)_{D \in \mathcal{D}}$ exists, where P' has power $\leqslant \lambda$. The $P' \cap D$ are dense in P' and P' is ccc. If $G' \subset P'$ is a filter intersecting every $P' \cap D$, then $G =_{\text{def}} \{q \in P \mid \exists p \in G'(p \leqslant q)\}$ the filter required. □

8.5.3 Consistency of Martin's axiom

Recall the following (see Exercises 204 and 205 p. 80):

Lemma 8.81 $G \subset P$ *is an* $\mathbf{M}$*-generic filter iff*

G1. $1 \in G$,

G2. $p \in G \wedge p \leqslant q \Rightarrow q \in G$,

G3'. $p, q \wedge G \Rightarrow \exists r(r \leqslant p, q)$,

G4'. G intersects every maximal antichain in $\mathbf{M}$.

Assume $P \subset Q$. We want a condition on P such that, if $G \subset Q$ is generic, then $P \cap G$ is generic over P (and, hence: $\mathbf{M}[P \cap G]$ is defined and $\subset \mathbf{M}[G]$: forcing extensions via Q are extensions of forcing extensions via P).

Suppose that A is a maximal antichain in P, but that $q \in Q$ is incompatible with every $p \in A$. Then if $G \ni q$ is generic, $P \cap G$ is not generic since it doesn't intersect $P \cap G$. Thus, a necessary condition is, that a maximal antichain $\subset P$ stays that way in Q. According to Lemma 8.83, this condition turns out to be sufficient as well.

Definition 8.82 Let $P \subset Q$. P is a *complete* suborder of Q, notation: $P \subset_c Q$, if every maximal antichain in P is also a maximal antichain in Q.

Examples.
1. If P is dense in Q, then $P \subset_c Q$.
2. $P \times \{1\} \subset_c P \times Q$. (N.B.: $P \cong P \times \{1\}$.)

250 ♣ Exercise. If $P \subset_c Q$, then P and Q have the same incompatibility notion. (If $p, q \in P$, then $p \perp q$ in P iff $p \perp q$ in Q.)

251 ♣ Exercise. Suppose $P \subset Q$. Show: $P \subset_c Q$ iff both (i) P and Q have the same incompatibility notion, and (ii) $\forall q \in Q \exists p \in P \forall p' \in P(p' \leqslant p \Rightarrow p' \sim q)$.

(It follows that $\subset_c$ is absolute.)

Lemma 8.83 *If* $\mathbf{M} \models P \subset_c Q$ *and* $G \subset Q$ *is* $\mathbf{M}$*-generic, then:*

1. $P \cap G$ *is* $\mathbf{M}$*-generic,*
2. *on* $\mathbf{M}^P$*, decoding w.r.t.* $P \cap G$ *is the same as decoding w.r.t.* G*,*
3. $\mathbf{M}[P \cap G] \subset \mathbf{M}[G]$*,*
4. *if* Φ *is restricted (or* Δ_1^{ZF}*),* $\tau_1, \ldots, \tau_k \in \mathbf{M}^P$ *and* $p \in P$*, then* $p \Vdash_P \Phi(\tau_1, \ldots, \tau_k) \Leftrightarrow p \Vdash_Q \Phi(\tau_1, \ldots, \tau_k)$,
5. *if* Q *is ccc, and* $p \in P$ Q*-forces that* τ *is a ccc poset (where* $\tau \in \mathbf{M}^P$*), then* p *also* P*-forces that* τ *is ccc.*

252 ♣ Exercise. Prove 8.83.

Definition 8.84 A sequence $\{P_\alpha \mid \alpha < \gamma\}$ (where γ is a limit) is *normal* if (i) $\alpha < \beta < \gamma \Rightarrow P_\alpha \subset_c P_\beta$, (ii) if $\beta < \gamma$ is a limit, then $P_\beta = \bigcup_{\alpha<\beta} P_\beta$.

Lemma 8.85 *1.* $P \subset_c Q \wedge Q \subset_c R \Rightarrow P \subset_c R$,

2. the union of a normal chain of posets is a complete extension of all elements of the chain.

Lemma 8.86 *Suppose that* $P \subset_c Q$*. Then* $\forall q \in Q \exists p \in P \forall p' \in P(p' \perp q \Rightarrow p' \perp p)$.

A condition $p \in P$ such that $\forall p' \in P(p' \perp q \Rightarrow p' \perp p)$ is called a (P-)*reduct* of q. Example: $(p, 1)$ is a $P \times \{1\}$-reduct of $(p, q) \in P \times Q$.

Proof. Choose $A \subset P$ maximal such that $A \cup \{q\}$ is an antichain. A is not a maximal antichain in Q. Thus, A is not a maximal antichain in P. Choose $p \in P$, $p \notin A$, s.t. $A \cup \{p\}$ is an antichain. This is the p required: assume that $p' \perp q$ but $p' \sim p$. Say, $r \leqslant p', p$. Since $P \subset_c Q$, we may assume that $r \in P$. We have that $r \perp q$ (if not, it would follow that $p' \sim q$). Also, r is incompatible with all elements of A (for, p has that property and $r \leqslant p$). Contradicting choice of A. □

Lemma 8.87 *The union of a normal chain of ccc posets is ccc.*

Proof. Suppose that $P = \bigcup_{\xi<\gamma} P_\xi$ (where $\{P_\xi \mid \xi < \gamma\}$ is a normal chain of ccc posets), and $A \subset P$ is an antichain of power ω_1.
1. $\mathrm{cf}(\gamma) = \omega$. Say, $\gamma = \bigcup_n \gamma_n$. Then uncountably many $p \in A$ occur in one P_{γ_n}.
2. $\mathrm{cf}(\gamma) \geqslant \omega_2$. Then for some $\xi < \gamma$, $A \subset P_\xi$.
3. $\mathrm{cf}(\gamma) = \omega_1$. Then P is the union of a normal chain of length ω_1.
So the result follows if we can show it for the case that:
4. $\gamma = \omega_1$.
Define $\rho : P \to \omega_1$ by $\rho(p) = \bigcap\{\xi \mid p \in P_\xi\}$. Note that $\rho(p)$ never is a limit ordinal.
By Lemma 8.86, if $\rho(p) > 0$, say, $\rho(p) = \xi + 1$, choose a reduct $f(p) \in P_\xi$ of p. Then

(a) $p \notin P_0 \Rightarrow \rho(f(p)) < \rho(p)$,

(b) for some integer $n = n(p)$, $f^n(p) \in P_0$,

(c) $\rho(q_1) < \rho(q_2) \wedge q_1 \perp q_2 \Rightarrow q_1 \perp f(q_2)$.

Write $A = \{q_\mu \mid \mu < \omega_1\}$. Since the $\rho(q)$ $(q \in A)$ must be unbounded in ω_1, we may as well assume they're all different.

For $q \in A$, define
$$S(q) =_{\mathrm{def}} \{\rho(f^i(q)) \mid 0 \leqslant i \leqslant n(q)\}.$$
Since every two $\rho(q)$ are different, every two $S(q)$ are different as well; and we may as well assume they form an uncountable Δ-system of n-element sets with a root r.

Let $\alpha = \max(r)$, $k = n - |r|$, and $a_\mu = S(q_\mu) - r$.

By a pigeon-hole principle, we may assume that
$$\mu < \sigma < \omega_1 \Rightarrow \max(a_\mu) < \min(a_\sigma).$$
Note that $\{f^k(q_\mu) \mid \mu < \omega_1\} \subset P_\alpha$ is an uncountable antichain (and contradiction). Indeed, if, say, $\rho(q_\mu) < \rho(q_\sigma)$, then, by $2k$ applications of (c) above, we find that $f^k(q_\mu) \perp f^k(q_\sigma)$. □

Lemma 8.88 *Suppose that $A \subset P$ is an antichain and that, for $q \in A$, σ_q is a P-name. Then a P-name π exists such that for all $q \in A$: $q \Vdash \pi = \sigma_q$.*

Proof. Put $\pi = \bigcup_{q\in A}\{(\tau, r) \mid \tau \in \mathrm{Dom}(\sigma_q) \wedge r \leqslant q \wedge r \Vdash \tau \in \sigma_q\}$. Suppose that $G \ni q$ is generic; we have to show that $\pi_G = (\sigma_q)_G$:
$\subset$: Say, $(\tau, r) \in \pi$, $r \in G$. Thus, for some $q' \in A$, we have that $\tau \in \mathrm{Dom}(\sigma_{q'})$, $r \leqslant q'$, $r \Vdash \tau \in \sigma_{q'}$. Hence, $q' \in G$, $q' = q$, and $r \Vdash \tau \in \sigma_q$.
$\supset$: Say, $\tau \in \mathrm{Dom}(\sigma_q)$, $p \in G$, $p \Vdash \tau \in \sigma_q$. Choose $r \in G$ such that $r \leqslant p, q$. Then $(\tau, r) \in \pi$. □

Lemma 8.89 (Maximum principle) $p \Vdash \exists x\Phi(x) \;\Rightarrow\; \exists\pi(p \Vdash \Phi(\pi))$.

Proof. Assume $p \Vdash \exists x\Phi(x)$ Then $D = \{q \mid q \leqslant p \;\wedge\; \exists\sigma(q \Vdash \Phi(\sigma))\}$ is dense below p. Choose a maximal antichain $A \subset D$. Then A is a maximal antichain $\subset \{q \mid q \leqslant p\}$ as well. For $q \in A$, choose σ_q s.t. $q \Vdash \Phi(\sigma_q)$. By the previous lemma, π exists s.t. $q \Vdash \Phi(\sigma_q) \wedge \pi = \sigma_q$ and, hence, $q \Vdash \Phi(\pi)$ $(q \in A)$. But then we have $p \Vdash \Phi(\pi)$ as well, since every generic $G \ni p$ intersects A. □

The next result (partly) generalizes the Product lemma 8.74 (p. 101) (part 3 is a bonus).

Theorem 8.90 (Product lemma, general version) *Let* $P \in \mathbf{M}$ *be a poset,* $\lambda \in \mathrm{OR} \cap \mathbf{M}$*,* $\preceq \in \mathbf{M}^P$*, and* $1 \Vdash [\preceq$ *partially orders* λ *with greatest element* $0]$*. Then* $Q \supset_c P$ *exists in* $\mathbf{M}$ *such that:*

1. *if* $G \subset Q$ *is* $\mathbf{M}$*-generic, there is an* $\mathbf{M}[P \cap G]$*-generic filter* $H \in \mathbf{M}[G]$ *over* $\preceq_G$ $(=\preceq_{P\cap G})$*,*
2. *in* $\mathbf{M}$*,* $|Q| = |P||\lambda|$*,*
3. *if* P *is ccc in* $\mathbf{M}$*, and* $1 \Vdash \preceq$ *is ccc, then* Q *is ccc in* $\mathbf{M}$*.*

Note that the assumption that the new poset orders an ordinal λ is inessential since, by AC, every poset is isomorphic to a poset of some ordinal.

Theorem 8.90 states, that every generic extension $\mathbf{M}[G]$ via Q includes a model obtained by *first*, forcing over P, obtaining $\mathbf{M}[P \cap G]$, *next*, forcing over $\preceq_G$, obtaining $\mathbf{M}[P \cap G][H] \subset \mathbf{M}[G]$.

In fact, more holds, as was the case in Lemma 8.74; but we have no need for that.

(In connection with 8.90.3, note that the statement that a product of ccc posets is ccc is independent in ZFC.)

253 ♣ Exercise. Formulate and prove the complete generalisation of Lemma 8.74.

Proof of Theorem 8.90.
Define the required poset Q on $P \times \lambda$, notation: $Q = P* \preceq$, by

$$(p, \xi) \leqslant (q, \delta) \equiv_{\mathrm{def}} p \leqslant q \;\wedge\; p \Vdash \xi \preceq \delta.$$

The map $p \mapsto (p, 0)$ is a complete embedding.

Note that, if Φ is restricted, and $\pi_1, \ldots, \pi_k \in \mathbf{M}^P$, then $(p, \xi) \Vdash_Q \Phi(\pi_1, \ldots, \pi_k)$ iff $(p, 0) \Vdash_Q \Phi(\pi_1, \ldots, \pi_k)$.
$(\Rightarrow:$ If $(p, 0) \nVdash_Q \Phi(\pi_1, \ldots, \pi_k)$, then, by Lemma 8.83.4, $p \nVdash_P \Phi(\pi_1, \ldots, \pi_k)$; say, $q \leqslant p$, $q \Vdash_P \neg\Phi(\pi_1, \ldots, \pi_k)$. Choose $G \ni (q, \xi)$ generic. Then $q \in P \cap G$, etc.)
8.90.1: Suppose that $G \subset Q$ is $\mathbf{M}$-generic. Put $H = \mathrm{Ran}(G)$. Leaving the fact that H is a filter as an exercise, we check $\mathbf{M}[P \cap G]$-genericity. (N.B.: $P \cap G = \mathrm{Dom}(G)$.)

Suppose that $D \in \mathbf{M}[P \cap G]$ is dense in $(\lambda, \preceq_G)$. Say, $p \in P \cap G$ forces $\mathbf{D}$ to be dense. Define $D^* = \{(q, \xi) \mid q \leqslant p \;\wedge\; q \Vdash \xi \in \mathbf{D}\}$.
Claim. D^* is dense below $(p, 0)$.
For, suppose $(r, \sigma) \leqslant (p, 0)$. Thus, r forces $\mathbf{D}$ is dense, i.e., it forces $\exists\xi \in \lambda(\xi \in \mathbf{D} \wedge \xi \preceq \sigma)$, and hence, some $q \leqslant r$ forces, for some $\xi \in \lambda$, that $(\xi \in \mathbf{D} \;\wedge\; \xi \preceq \sigma)$, i.e.: $(q, \xi) \leqslant (r, \sigma)$ and $(q, \xi) \in D^*$.

Therefore, G intersects D^*, say, $(q, \xi) \in G \cap D^*$. Thus, $q \Vdash \xi \in \mathbf{D}$ and $\xi \in D \cap H$.

The last part of the proof is rather surprising.

8.90.3: Suppose that, in $\mathbf{M}$, P is ccc and 1 forces $\preceq$ to be ccc. Let $\{(p_\mu, \xi_\mu) \mid \mu < \omega_1\}$ be an antichain in Q. If $\xi_\mu = \xi_\sigma$, then $p_\mu \perp p_\sigma$; P is ccc, and so for every ξ there are at most ω μ s.t. $\xi_\mu = \xi$. Therefore, we may as well suppose that every two ξ_μ are different.

By a whim of coding, $\pi =_{\text{def}} \{(\xi_\mu, p_\mu) \mid \mu < \omega_1\}$ happens to be a P-name; clearly, 1 forces that $\pi \subset \lambda$.

Claim. $1 \Vdash \pi$ is an $\preceq$-antichain.

For, let $G' \subset P$ be $\mathbf{M}$-generic, let $\xi_\mu, \xi_\sigma \in \pi_{G'}$ —thus, $p_\mu, p_\sigma \in G'$. Now if $\rho \preceq_{G'} \xi_\mu, \xi_\sigma$, then some $p \in G'$ forces $\rho \preceq \xi_\mu, \xi_\sigma$. Choose $q \leqslant p, p_\mu, p_\sigma$ in G'. Then $(q, \rho) \leqslant (p_\mu, \xi_\mu), (p_\sigma, \xi_\sigma)$: a contradiction.

By this claim, 1 forces that π is countable; say (Maximum principle), that $\mathbf{f}$ is a surjection : $\omega \to \pi$. Since P is ccc, every set $A_n =_{\text{def}} \{\xi \mid \exists p(p \Vdash \mathbf{f}(n) = \xi)\}$ is countable. Thus, $A =_{\text{def}} \bigcup_n A_n$ is countable. But, $A \supset \{\xi_\mu \mid \mu < \omega_1\}$ (for, $p_\mu \Vdash \xi_\mu \in \pi$; hence $p \leqslant p_\mu$ and n exist s.t. $p \Vdash \mathbf{f}(n) = \xi_\mu$) —a contradiction. □

254 ♣ Exercise. Complete the details of the proof of Theorem 8.90. I.e., check: Q is a poset, $P \subset_c Q$ via the embedding $p \mapsto (p, 0)$, H is a filter.

Theorem 8.91 *If, in* $\mathbf{M}$*:* $\omega_1 < \text{cf}(\kappa) = \kappa$ *and* $\forall \lambda < \kappa (2^\lambda \leqslant \kappa)$*, then a* ccc$^{\mathbf{M}}$ *poset* $P \in \mathbf{M}$ *exists such that, whenever* $G \subset P$ *is* $\mathbf{M}$*-generic, we have in* $\mathbf{M}[G]$ *that:* $\forall \lambda < \kappa (2^\lambda \leqslant \kappa)$*, and for every ccc poset over some* $\lambda < \kappa$ *and every collection of less that* κ *dense sets there is a filter intersecting all of them.*

Clearly, in $\mathbf{M}[G]$, we have that $\forall \lambda < \kappa (\text{MA}(\lambda))$: by Lemma 8.80, the restriction to posets of cardinality $< \kappa$ is irrelevant, and every such poset is $\cong$ to one over some $\lambda < \kappa$. By Remark 8.79.3 (p. 104), it follows that $\kappa \leqslant 2^\omega$, and hence, $2^\omega = \kappa$ and so we have that MA holds. Thus, MA implies nothing about the value of 2^ω, except that it be regular.

The conditions $\text{cf}(\kappa) = \kappa$ and $\forall \lambda < \kappa (2^\lambda \leqslant \kappa)$ are necessary: MA implies that 2^ω is regular (so we need κ to be regular if we want to preserve regularity and get $2^\omega = \kappa$); and it also implies (8.79.6/7) $\omega \leqslant \lambda < 2^\omega \Rightarrow 2^\lambda = 2^\omega$ (so if we had $\kappa < 2^\lambda$ in $\mathbf{M}$ for some $\lambda < \kappa$, this would be the case in $\mathbf{M}[G]$ as well, since 2^λ can only grow).

Proof. The required poset P will be the limit of a normal chain of ccc posets P_ξ $(\xi < \kappa)$ of power $< \kappa$, that will be constructed (in $\mathbf{M}$) in such a way that, whenever $G \subset P$ is $\mathbf{M}$-generic, and, in $\mathbf{M}[G]$, R is a ccc partial order of some $\lambda < \kappa$ (with greatest element 0) and $\mathcal{D} <_1 \kappa$ is a collection of R-dense sets, then $\alpha < \kappa$ and a P_α-name τ exist such that

1. $R = \tau_G = \tau_{(P_\alpha \cap G)} \in \mathbf{M}[P_\alpha \cap G]$ and $\mathcal{D} \subset \mathbf{M}[P_\alpha \cap G]$,
2. $1_\alpha \Vdash_{P_\alpha} \tau$ is a ccc poset over λ (with greatest element 0),
3. $P_{\alpha+1} = P_\alpha * \tau$ (cf. the proof of Theorem 8.90).

By Theorem 8.90.1, $\mathbf{M}[P_{\alpha+1} \cap G]$ (and hence $\mathbf{M}[G]$) has a $\mathbf{M}[P_\alpha \cap G]$-generic filter over R that (since $\mathcal{D} \subset \mathbf{M}[P_\alpha \cap G]$), intersects every $D \in \mathcal{D}$.

Intuitively, $\mathbf{M}[G]$ arises as a limit of a chain of forcing-extensions of length κ, that eventually uses *every* ccc poset of the resulting $\mathbf{M}[G]$. But of course, *in fact*, $\mathbf{M}[G]$ is constructed as a single forcing extension, and it is the poset P that is constructed using a chain.

The construction of this chain uses a surjection $h : \kappa \to \kappa \times \kappa$ with the property that $h(\alpha) = (\beta, \gamma) \Rightarrow \beta \leqslant \alpha$.

This is used in the following way. At every stage β of the intuitive construction of $\mathbf{M}[G]$, we can force over one poset only. That will mean that we have to postpone, to later stages $\alpha > \beta$, forcing over most of the posets that are available already at stage β (i.e., are member of $\mathbf{M}[P_\beta \cap G]$).

Whenever a stage β is reached, we fix (in $\mathbf{M}$) an enumeration $\{(\lambda_{\beta\gamma}, \pi_{\beta\gamma}) \mid \gamma < \kappa\}$ of all pairs (λ, π) where $\lambda < \kappa$ and π is a nice P_β-name for a subset of $\lambda \times \lambda$. (Check that, if $P_\beta <_1 \kappa$ is ccc, there are not more than κ nice names.)

Now the meaning of $h(\alpha) = (\beta, \gamma)$ will be that, at stage α, we'll use the γ-th poset $(\pi_{\beta\gamma})_G$ that has become available at stage β (provided this is a poset indeed) to construct the next extension.

The construction of the P_α is as follows. P_0 is an arbitrary ccc poset over some $\lambda < \kappa$; at limits, we take unions. Finally, suppose that P_α has been constructed. To construct $P_{\alpha+1}$, assume that $h(\alpha) = (\beta, \gamma)$. Thus, $\beta \leqslant \alpha$, and P_α, $\pi = \pi_{\beta\gamma}$ and $\lambda = \lambda_{\beta\gamma}$ are known.

We'd like to define $P_\alpha =_{\text{def}} P_\alpha * \pi$ (see the proof of Theorem 8.90), but we don't know whether the necessary condition: that $1_\alpha \Vdash$(π is a ccc poset), holds. However, by logic alone, $1_\alpha \Vdash \exists x[x$ is a ccc poset on $\lambda \wedge$ (π is a ccc poset on $\lambda \to x = \pi$)]; and hence (by the Maximum principle), for some $\tau = \tau_\alpha$ we have that $1_\alpha \Vdash$ [τ is a ccc poset on $\lambda \wedge$ (π is a ccc poset on $\lambda \to \tau = \pi$)]. Thus, we put $P_\alpha =_{\text{def}} P_\alpha * \tau$. This completes the construction of the P_α and, hence, of P.

To see that this accomplishes what is required, assume that $G \subset P$ is $\mathbf{M}$-generic, and that, in $\mathbf{M}[G]$, R is a ccc partial order of $\lambda < \kappa$ (with greatest element 0) and $\mathcal{D} <_1 \kappa$ is a collection of R-dense sets $\subset \lambda$.

Choose a nice P-name π for R. Some $p \in G$ P-forces that π is a ccc poset of λ.

Note that, for some $\beta < \kappa$, π is a P_β-name (and, hence, $R \in \mathbf{M}[G \cap P_\beta]$), since P is ccc and κ is regular. By a similar argument (use nice names for elements of $\mathcal{D}$), $\mathcal{D} \subset \mathbf{M}[G \cap P_\beta]$ for some $\beta < \kappa$. Thus, assume that $\beta < \kappa$ is chosen in such a way, that $p \in P_\beta$, π is a P_β-name, and $\mathcal{D} \subset \mathbf{M}[G \cap P_\beta]$.

Say, $\pi = \pi_{\beta\gamma}$, $\lambda = \lambda_{\beta\gamma}$, $h(\alpha) = (\beta, \gamma)$ (and so, $\beta \leqslant \alpha$).

By Lemma 8.83.5 (p. 105), p also P_α-forces that π is a ccc poset. Therefore, by choice of $\tau = \tau_\alpha$, $p \Vdash_{P_\alpha} \tau = \pi$, and hence $R = \pi_G = \tau_G$.

Preservation of the condition $\forall \lambda < \kappa (2^\lambda \leqslant \kappa)$ follows by a nice names-argument. □

Solutions to Selected Exercises

Chapter 2

5. Deduce the Pairing Axiom from the other axioms.
Solution. First, we need (existence of) a set A with at least two (different) elements (such as $\{\emptyset, \{\emptyset\}\}$ — but ask yourself: what does the phrase "$\{\emptyset, \{\emptyset\}\}$ exists as a set" mean?).

Two ways to get such a set A:

(i) Existence of $\emptyset$ follows from Separation ($\emptyset = \{x \in a \mid x \neq x\}$, where a is any set obtained from Axiom 0) or from Infinity.

Note that $\wp(\emptyset) \neq \emptyset$. For, $\emptyset \notin \emptyset$, whereas $\emptyset \in \wp(\emptyset)$ (check this).

Now, note that $\emptyset, \wp(\emptyset) \in A := \wp\wp(\emptyset)$.

(Of course, you can show that, in fact, $\wp(\wp(\emptyset)) = \{\emptyset, \{\emptyset\}\}$.)

(ii) Alternatively, if A is obtained from Infinity, then $\emptyset$ and $\emptyset \cup \{\emptyset\} = \{\emptyset\}$ are different elements of A.

Now let a and b be any sets. Define F on A by $F(\emptyset) = a$ and $F(x) = b$ for $x \neq \emptyset$. (Note that F is first-order definable: $F(x) = y :\equiv (x = \emptyset \wedge y = a) \vee (x \neq \emptyset \wedge y = b)$.) Then by Substitution $\{a, b\} = \{F(x) \mid x \in A\}$ exists as a set. □

6. Deduce the Separation Axiom from the other axioms.
Solution. Use Infinity to see that $\emptyset$ exists. Let a be a set and E be a property. If $\{x \in a \mid E(x)\} = \emptyset$, there remains nothing to prove. Otherwise, pick $b \in a$. Define F on a by letting $F(x) = x$ if $E(x)$ and $F(x) = b$ otherwise. (Note that the condition $F(x) = y$ is first-order definable.) Now, $\{x \in a \mid E(x)\} = \{F(x) \mid x \in a\}$ exists as a set by Substitution.
Alternatively, using Pairing, directly define G on a by letting $F(x) = \{x\}$ if $E(x)$ and $F(x) = \emptyset$ otherwise. Now, $\bigcup\{G(x) \mid x \in a\}$ exists as a set by Substitution and Sumset. However, to see that, indeed, $\{x \in a \mid E(x)\} = \bigcup\{G(x) \mid x \in a\}$, again requires Pairing! □

7. Show that for any sets A and B, the Cartesian product $A \times B := \{(a, b) \mid a \in A \wedge b \in B\}$ exists as a set.
Solution. Using Powerset and Separation: $A \times B = \{p \in \wp(\wp(A \cup B)) \mid \exists a \in A \exists b \in B (p = (a, b))\}$ (noting that $A \times B \subset \wp\wp(A \cup B)$).
Alternatively, using Substitution: By Substitution, for each $b \in B$, $A \times \{b\} = \{(a, b) \mid a \in A\}$ is a set. Define G on B by: $G(b) := A \times \{b\}$. By Substitution and Sumsets, $A \times B = \bigcup\{G(b) \mid b \in B\}$ exists as a set. □

8. Show:

1. $\{p, q\} = \{p, r\} \;\Rightarrow\; q = r$,
2. $(x, y) = (a, b) \;\Rightarrow\; x = a \wedge y = b$.

Solution. 1. Assume $\{p, q\} = \{p, r\}$. Since $q \in \{p, q\}$, it follows that $q \in \{p, r\}$; so $q = p$ or $q = r$. As $q = r$ has to be proved, assume that $q = p$. A similar argument shows that you can assume that $r = p$. So, after all, $q = p = r$.

2. Assume that $(x, y) = (a, b)$. Since $\{x\} \in (x, y)$, it follows that $\{x\} \in (a, b)$; so (i) $\{x\} = \{a\}$ or (ii) $\{x\} = \{a, b\}$. If (i) $\{x\} = \{a\}$, then, applying 1 to (x, y) and (a, b), you obtain that $\{x, y\} = \{a, b\}$. From $\{x\} = \{a\}$ it follows that $x = a$. So by a second application of 1 it follows that $y = b$. If (ii) $\{x\} = \{a, b\}$, then $x = a = b$, $(a, b) = \{\{b\}\}$, $\{x, y\} = \{b\}$, and $y = b$. □

15. Show that the following classes are proper:

- $\mathcal{R} := \{x \mid x \notin x\}$,
- $\mathbf{V} = \{x \mid x = x\}$,
- $\mathcal{Q}_n := \{x \mid \neg\exists x_1, \ldots, x_n(x_1 \in x \wedge x_2 \in x_1 \wedge \cdots \wedge x_n \in x_{n-1} \wedge x \in x_n)\}$,
- $\mathcal{G} := \{x \mid \forall a(x \in a \Rightarrow \exists y \in a \forall z \in a(z \notin y))\}$.

Solution. Assume that $\mathcal{R}$ is a set. Then $\mathcal{R} \in \mathcal{R}$ iff $\mathcal{R} \notin \mathcal{R}$, a (propositional) contradiction.

Since $\mathcal{R} \subset \mathbf{V}$, if $\mathbf{V}$ is a set, then, by Separation, $\mathcal{R}$ also is a set.

Suppose that $\mathcal{Q}_n$ is a set. If $\mathcal{Q}_n \in \mathcal{Q}_n$, then $\mathcal{Q}_n \in \mathcal{Q}_n \in \cdots \in \mathcal{Q}_n \in \mathcal{Q}_n$. Thus, $\mathcal{Q}_n$ does not satisfy its defining property, showing that, after all, $\mathcal{Q}_n \notin \mathcal{Q}_n$. But then, $x_1, \ldots, x_n$ must exist such that $\mathcal{Q}_n \in x_n \in x_{n-1} \in \cdots \in x_1 \in \mathcal{Q}_n$. Putting this a little differently, $x_1 \in \mathcal{Q}_n \in x_n \in x_{n-1} \in \cdots \in x_1$. Consequently, by definition of $\mathcal{Q}_n$, $x_1 \notin \mathcal{Q}_n$, a contradiction.

Finally, assume that $\mathcal{G}$ is a set. Then $\mathcal{G} \in \mathcal{G}$. For, suppose $\mathcal{G} \in a$; we want $y \in a$ s.t. $y \cap a = \emptyset$. If $\mathcal{G} \cap a = \emptyset$, we're done. If $\mathcal{G} \cap a \neq \emptyset$, say: $x \in \mathcal{G} \cap a$, then since $x \in \mathcal{G}$ and $x \in a$, there is such a y.

But now $\mathcal{G} \in \{\mathcal{G}\}$, whereas $\{\mathcal{G}\}$ isn't disjoint with one of its elements. □

17. Let A be a set. Show that the following subsets of A do not belong to A.

- $\mathcal{R}_A := \{x \in A \mid x \notin x\}$,
- $\mathcal{Q}_{n,A} := \{x \in A \mid \neg\exists x_1, \ldots, x_n(x_1 \in x \wedge x_2 \in x_1 \wedge \cdots \wedge x_n \in x_{n-1} \wedge x \in x_n)\}$,
- $\mathcal{G}_A := \{x \in A \mid \forall a(x \in a \Rightarrow \exists y \in a \forall z \in a(z \notin y))\}$.

Solution. The results of 15 refer to the system $(\mathbf{V}, \in)$ (with respect to the definition of $\mathcal{G}$, $\forall a$ quantifies over subsets of $\mathbf{V}$) about which no assumption has to be made (the arguments do not use ZF axioms). Therefore, these results apply to arbitrary systems $(\mathbf{U}, \varepsilon)$, in particular, to systems of the form $(A, \in)$. □

Chapter 3

22. Are the following (always) true?

1. If every $x \in A$ is transitive, then so is $\bigcup A$.
2. If A is transitive, then so is $\bigcup A$.
3. If $\bigcup A$ is transitive, then so is A.

Solution. 1. True. If $x \in y \in \bigcup A$, then some $z \in A$ exists s.t. $y \in z$. Since z is transitive, $x \in z$; and so $x \in \bigcup A$.

2. True. Similar argument.

3. False. Consider $A := \{\{\emptyset\}\}$ (this is the simplest counter-example).

34.

1. Give an example of a closed set different from ω.
2. Show that *every* set is subset of a closed set.
3. Show that every set is subset of a *least* closed set.
4. Suppose that o is a set and s an operation on sets. Show that a set a exists for which $o \in a$ and such that for all $x \in a$, $s(x) \in a$.

Solution. 3. Let a be an arbitrary set. Recursively, define F on ω by $F(0) = a \cup \{0\}$ and $F(S(n)) = F(n) \cup \{\mathrm{S}(x) \mid x \in F(n)\}$. Then (show that) $\bigcup_n F(n)$ is the least inductive set $\supset a$. □

36. Prove Lemma 3.19.1–3. Prove Lemma 3.19.4, and do *not* use 3.18, but use 3.19.1–3.
Solution.

1. $a = \mathrm{TC}(a, 0) \subset \mathrm{TC}(a)$.

2. If $x \in y \in \mathrm{TC}(a)$, then, for some $n \in \omega$, $y \in \mathrm{TC}(a, n)$. Thus, $x \in \bigcup \mathrm{TC}(a, n) = \mathrm{TC}(a, \mathrm{S}(n)) \subset \mathrm{TC}(a)$.

3. If $b \supset a$ is transitive then, by induction on n, $\mathrm{TC}(a, n) \subset b$. Thus, $\mathrm{TC}(a) \subset b$.

4. $\supset$: First, $a \subset \mathrm{TC}(a)$. Next, if $b \in a$, then $b \in \mathrm{TC}(a)$ (since $a \subset \mathrm{TC}(a)$), $b \subset \mathrm{TC}(a)$ (since $\mathrm{TC}(a)$ is transitive), and $\mathrm{TC}(b) \subset \mathrm{TC}(a)$ (by property 3).

$\subset$: By property 3, it suffices to show that $a \cup \bigcup_{b \in a} \mathrm{TC}(b)$ is a transitive superset of a. Transitivity: if $x \in y \in a \cup \bigcup_{b \in a} \mathrm{TC}(b)$, then $y \in a$, or $b \in a$ exists such that $y \in \mathrm{TC}(b)$. In the first case, $x \in \mathrm{TC}(y) \subset a \cup \bigcup_{b \in a} \mathrm{TC}(b)$. In the second, $x \in \mathrm{TC}(b) \subset a \cup \bigcup_{b \in a} \mathrm{TC}(b)$. □

40. A relation R is *confluent* if $\forall a \forall b \forall c(aRb \wedge aRc \Rightarrow \exists d(bRd \wedge cRd))$. Show that if R is confluent, then so is $R^\star$.

Solution. Assume that R is confluent. First, here is a

Lemma. $\forall a \forall b \forall c(aRb \wedge a\, R^\star\, c \Rightarrow \exists d(b\, R^\star\, d \wedge cRd))$.

Proof. Assume that aRb and $aR \cdots Rc$. Induction w.r.t. the length of the sequence $aR \cdots Rc$. Length 1, i.e.: aRc: R is confluent. Length $n+1$, i.e., $aR \cdots Rc'Rc$: by IH, d' exists s.t. $c'Rd'$ and bRd'. Apply confluency to c', c, d' to obtain d such that cRd and $d'Rd$; then also $b\, R^\star\, d'$. □

Now, suppose that $aR \cdots Rb$ and $a\, R^\star\, c$. Induction w.r.t. the length of the sequence $aR \cdots Rb$. Length 1, i.e., aRb: apply lemma. Length $n+1$, i.e.: $aR \cdots Rb'Rb$: by IH, there exists d' s.t. $b'\, R^\star\, d'$ and $c\, R^\star\, d'$. Apply the lemma to b', d', b to obtain d such that $b\, R^\star\, d$ and $d'Rd$. Then also $c\, R^\star\, d$.

44. Prove Theorem 3.27.

Hint. Do *not* use Theorem 3.24. Show that $n < m \Rightarrow H{\Uparrow}n \subset H{\Uparrow}m$. Show: if $H(X) \subset X$, then, for all n, $H{\Uparrow}n \subset X$. Finally, show that $H(H{\Uparrow}\omega) \subset H{\Uparrow}\omega$. (For this, you will need the fact that if Y is finite and $Y \subset \bigcup_{n \in \omega} H{\Uparrow}n$, then for some $m \in \omega$, $Y \subset H{\Uparrow}m$. This is shown by induction w.r.t. the *number of elements* of Y, cf. Definition 3.15, p.15.)

Solution. 1. Proof that for all n, $\forall m(n < m \Rightarrow H{\Uparrow}n \subset H{\Uparrow}m)$, by induction w.r.t. n:
Basis, $n = 0$. If $0 < m$, then $H{\Uparrow}n = \emptyset \subset H{\Uparrow}m$ is obvious.
Induction step. Assume induction hypothesis $\forall m(n < m \Rightarrow H{\Uparrow}n \subset H{\Uparrow}m)$. Now, let $n+1 < m$. Then m' exists such that $m'+1 = m$, and $n < m'$. By IH, $H{\Uparrow}n \subset H{\Uparrow}m'$. Thus, $H{\Uparrow}n+1 = H(H{\Uparrow}n) \subset H(H{\Uparrow}m') = H{\Uparrow}m'+1 = H{\Uparrow}m$.
2. Suppose that $H(X) \subset X$. By induction on n, it follows that $H{\Uparrow}n \subset X$:
Basis $n = 0$: $H{\Uparrow}0 = \emptyset \subset X$ is obvious.
Induction step: assume (IH) $H{\Uparrow}n \subset X$. Then by monotonicity, $H{\Uparrow}n+1 = H(H{\Uparrow}n) \subset H(X) \subset X$.
3. If $Y \subset H{\Uparrow}\omega = \bigcup_n H{\Uparrow}n$ is finite, then n exists s.t. $Y \subset H{\Uparrow}n$: induction w.r.t. nr of elements of Y.
Basis, $Y = \emptyset$. Then $Y \subset \emptyset = H{\Uparrow}0$.
Induction step. IH: for n-element Y, the statement holds. Now let $Y \subset \bigcup_n H{\Uparrow}n$ have $n+1$ elements. For instance, $Y = Y' \cup \{y\}$, where Y' has n elements. By IH, n_1 exists s.t. $Y \subset H{\Uparrow}n_1$. Furthermore, n_2 exists s.t. $y \in H{\Uparrow}n_2$. Let $m = \max(n_1, n_2)$. Then clearly (by 1), $Y \subset H{\Uparrow}m$.
4. $H(H{\Uparrow}\omega) \subset H{\Uparrow}\omega$:
Assume that $a \in H(H{\Uparrow}\omega)$. By finiteness, a finite $Y \subset H{\Uparrow}\omega$ exists s.t. $a \in H(Y)$. By 3, suppose that $Y \subset H{\Uparrow}n$. Then $a \in H(Y) \subset H(H{\Uparrow}n) = H{\Uparrow}n+1 \subset H{\Uparrow}\omega$. □

Chapter 4

58. Show:

1. $\alpha \leq \beta \Leftrightarrow \alpha \subset \beta$,
2. if K is a non-empty class of ordinals, then $\bigcap K$ is the least element of K,
3. if A is a *set* of ordinals, then $\bigcup A$ is an ordinal that is the *sup* of A (the least ordinal $\geq$ every $\alpha \in A$).

2. If $\alpha \in K$, then obviously $\bigcap K \leq \alpha$. To see that $\bigcap K \in K$: let $\beta \in K$ be the least element. Then again $\bigcap K \leq \beta$, and $\beta \leq \bigcap K$ is trivial.

61. Assume that $(A, \prec)$ is a well-ordering and $B \subset A$.
Show that $\mathrm{type}(B, \prec) \leqslant \mathrm{type}(A, \prec)$.

Solution. Again, Suppose that α and β are the order types of A and B. If $\beta \not\leq \alpha$, then $\alpha < \beta$. Let $f : \beta \to B$ and $g : A \to \alpha$ be order-isomorphisms. Then $gf : \beta \to \alpha$ is an order-preserving injection such that $gf(\alpha) < \alpha$. This contradicts Lemma 4.8 (p. 27).

70. Show that the *single* recursion equation $H{\uparrow}\alpha = \bigcup_{\xi<\alpha} H(H{\uparrow}\xi)$ defines the same operation as the one defined in Definition 4.14 by *three* equations. (And, of course, $H{\downarrow}\alpha = \bigcap_{\xi<\alpha} H(H{\downarrow}\xi)$ is a single equation defining the greatest fixed point hierarchy — cf. Exercise 72.)
Solution.
Claim: $H{\uparrow}\alpha \subset H{\uparrow}(\alpha+1)$.
Induction. Obviously, $H{\uparrow}0 \subset H{\uparrow}1$. And if $H{\uparrow}\alpha \subset H{\uparrow}(\alpha+1)$, then, by monotonicity, $H{\uparrow}(\alpha+1) \subset H{\uparrow}(\alpha+2)$. Finally, if γ is a limit, then, if $\xi < \gamma$, $H{\uparrow}\xi \subset \bigcup_{\xi<\gamma} H{\uparrow}\xi$, hence, $H(H{\uparrow}\xi) \subset H(\bigcup_{\xi<\gamma} H{\uparrow}\xi)$; and so $H{\uparrow}\gamma = \bigcup_{\xi<\gamma} H{\uparrow}\xi \subset \bigcup_{\xi<\gamma} H(H{\uparrow}\xi) \subset H(\bigcup_{\xi<\gamma} H{\uparrow}\xi) = H{\uparrow}(\gamma+1)$.

Now:

$H{\uparrow}0 = \emptyset = \bigcup_{\xi<0} H(H{\uparrow}\xi)$;

$H{\uparrow}(\alpha+1) = H(H{\uparrow}\alpha) = H(H{\uparrow}\alpha) \cup H{\uparrow}\alpha$ (since $H{\uparrow}\alpha \subset H(H{\uparrow}\alpha)$) $= H(H{\uparrow}\alpha) \cup \bigcup_{\xi<\alpha} H(H{\uparrow}\xi)$ (by IH) $= \bigcup_{\xi<\alpha+1} H(H{\uparrow}\xi)$;

$H{\uparrow}\gamma = \bigcup_{\xi<\gamma} H{\uparrow}\xi = \bigcup_{\xi<\gamma} H(H{\uparrow}\xi)$ (since $H{\uparrow}\xi \subset H{\uparrow}(\xi+1)$).

73. Show that $\mathrm{V}_\alpha = \bigcup_{\xi<\alpha} \wp(\mathrm{V}_\xi)$. (This is a *single* recursion equation defining the sequence $\{\mathrm{V}_\xi\}_\xi$. Cf. Exercise 70.)

Solution. Needed: Exercise 75.1: every V_α is transitive. Induction w.r.t. α.
Basis: $\alpha = 0$. Trivial.
Successor-step. Suppose that $\mathrm{V}_\alpha = \bigcup_{\xi<\alpha} \wp(\mathrm{V}_\xi)$.
Then $\mathrm{V}_{\alpha+1} = \wp(\mathrm{V}_\alpha) = \mathrm{V}_\alpha \cup \wp(\mathrm{V}_\alpha) = \bigcup_{\xi<\alpha} \wp(\mathrm{V}_\xi) \cup \wp(\mathrm{V}_\alpha) = \bigcup_{\xi<\alpha+1} \wp(\mathrm{V}_\xi)$.
Limit step. Suppose that γ is a limit and for all $\alpha < \gamma$ $\mathrm{V}_\alpha = \bigcup_{\xi<\alpha} \wp(\mathrm{V}_\xi)$.
Then $\mathrm{V}_\gamma = \bigcup_{\alpha<\gamma} \mathrm{V}_\alpha = \bigcup_{\alpha<\gamma} \wp(\mathrm{V}_\alpha)$. For the last identity, suppose $\alpha < \gamma$. Then: ($\subset$): $\mathrm{V}_\alpha \subset \wp(\mathrm{V}_\alpha)$; ($\supset$): $\wp(\mathrm{V}_\alpha) = \mathrm{V}_{\alpha+1}$ and $\alpha+1 < \gamma$.

75. Show:

1. *Every* V_α *is transitive,*
2. $x \subset y \in \mathrm{V}_\alpha \;\Rightarrow\; x \in \mathrm{V}_\alpha$,
3. $\alpha < \beta \;\Rightarrow\; \mathrm{V}_\alpha \in \mathrm{V}_\beta$; $\alpha \leq \beta \;\Rightarrow\; \mathrm{V}_\alpha \subset \mathrm{V}_\beta$,
4. $\alpha \subset \mathrm{V}_\alpha$; $\alpha \notin \mathrm{V}_\alpha$; $\alpha = \mathrm{OR} \cap \mathrm{V}_\alpha$,
5. $\mathrm{OR} \cap (\mathrm{V}_{\alpha+1} - \mathrm{V}_\alpha) = \{\alpha\}$.

Solution. Using the single recursion equation for the V_α:

1. Suppose that $x \in y \in \mathrm{V}_\alpha$. Then for some $\beta < \alpha$, $y \subset \mathrm{V}_\beta$; hence, $x \in \mathrm{V}_\beta$, by IH $x \subset \mathrm{V}_\beta$, and $x \in \mathrm{V}_\alpha$.

2. Suppose that $x \subset y \in \mathrm{V}_\alpha$. Then for some $\beta < \alpha$, $y \subset \mathrm{V}_\beta$, thus, $x \subset \mathrm{V}_\beta$, and $x \in \mathrm{V}_\alpha$.

3. Assume $\alpha < \beta$. $\mathrm{V}_\alpha \subset \mathrm{V}_\alpha$,

4. (i) If $\beta < \alpha$, then (since by IH, $\beta \subset \mathrm{V}_\beta$), $\beta \in \mathrm{V}_\alpha$. Thus, $\alpha \subset \mathrm{V}_\alpha$.

(ii) If $\alpha \in \mathrm{V}_\alpha$ then, for some $\beta < \alpha$, $\alpha \subset \mathrm{V}_\beta$. Thus, $\beta \in \mathrm{V}_\beta$, contradicting IH.

The rest follows immediately.

Chapter 5

99. Show: (i) $<$ is a well-ordering of $\mathrm{OR} \times \mathrm{OR}$, (ii) for every γ, $\gamma \times \gamma$ is a $<$-initial of $\mathrm{OR} \times \mathrm{OR}$ and (iii) for all α, $\omega_\alpha \times \omega_\alpha$ is well-ordered by $<$ in type ω_α.
Solution. (i) and (ii) are straightforward.

(iii) This is obvious for $\alpha = 0$. For $\alpha > 0$: Let Γ be the order-isomorphism from $\mathrm{OR} \times \mathrm{OR}$ onto OR. We want to show that, for every α, $\Gamma(\omega_\alpha, \omega_\alpha) = \omega_\alpha$. Suppose (IH) this true for $\alpha' < \alpha$, but $\Gamma(\omega_\alpha, \omega_\alpha) > \omega_\alpha$; say, $\beta, \gamma < \omega_\alpha$ and $\Gamma(\beta, \gamma) = \omega_\alpha$. Since $\alpha > 0$, an *infinite* $\delta < \omega_\alpha$ exists such that $\beta, \gamma \in \delta \times \delta$. By IH (for the α' s.t. $\omega_{\alpha'} = \delta$), $\delta \times \delta =_1 \delta$. Now, $\omega_\alpha = \Gamma(\beta, \gamma) \leqslant_1 \delta \times \delta =_1 \delta <_1 \omega_\alpha$, a contradiction.

103. (AC) Show: if A is infinite, then $\omega \leq_1 A$.
Show *without* AC that: if A is infinite, then $\omega \leq_1 \wp(\wp(A))$.
Solution. (i) Let j be a choice function for $\wp(A)$. Recursively, define $f : \omega \to A$ by $f(n) = j(A - \{f(m) \mid m < n\})$ as long as $A - \{f(m) \mid m < n\}) \neq \emptyset$; $f(n) = A$ otherwise. Obviously, since A is infinite, if $f|n$ is an injection then $A - \{f(m) \mid m < n\}) \neq \emptyset$. By induction, show that for all n, $f|n$ is an injection. Thus, f is an injection as well.

(ii) Show that for all n, A has an n-element subset. The required injection maps n to $\{B \subset A \mid |B| = n\}$.

108. The *Teichmüller-Tukey Lemma* is the following statement.

> *Suppose that $A \subset \wp(X)$ is such that for all $Y \subset X$, Y is in A iff every finite subset of Y is in A. Then A has a ($\subset$-) maximal element.*

Show that this is equivalent with Zorn's Lemma.

Solution.
Zorn $\Rightarrow$ TT:
Suppose that A is as in the TT Lemma. For A to have a maximal element, by Zorn, it suffices to show that it is closed under unions of chains. Thus, suppose that $K \subset A$ is a chain. In order that $\bigcup K \in A$, it suffices to show that every finite subset is in A. Thus, suppose that $C \subset \bigcup K$ is finite. Then for some $Y \in K$, we have that $C \subset Y$. Therefore, $C \in A$.

TT $\Rightarrow$ Zorn:
Let P be a non-empty poset in which chains have upper bounds. Let A be the set of all chains of P. Then A satisfies the TT condition: (i) a finite subset of a chain is a chain, and (ii) if every finite subset of $K \subset A$ is a chain, then K is a chain (if $a, b \in K$, then $\{a, b\}$ is a finite subset, hence $a \leqslant b$ or $b \leqslant a$). By TT, A has a maximal element, which is a maximal chain of P. An upper bound of this chain is maximal in P.

Chapter 6

122.

1. Suppose that $X \subset \alpha$ is cofinal in α. Show that α has a cofinal subset Y of type $\mathrm{cf}(\alpha)$ such that $Y \subset X$.
2. Show: if α and β have cofinal subsets of the same type, then $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$.
3. Show: if α is a limit, then $\mathrm{cf}(\omega_\alpha) = \mathrm{cf}(\alpha)$.

Solution.
1. Suppose that $f : \mathrm{cf}(\alpha) \to \alpha$ has $\mathrm{Ran}(f)$ cofinal in α. Define $g : \mathrm{cf}(\alpha) \to X$ by $g(\xi) = \bigcap\{\delta \in X \mid f(\xi) \leq \delta\}$. Clearly, $\mathrm{Ran}(g)$ is cofinal in α. Applying Lemma 6.26 (p. 47) yields an order-preserving function $h \subset g$ with $Y := \mathrm{Ran}(h) \subset X$ cofinal in α. Now $\mathrm{cf}(\alpha) \leq \mathrm{type}(Y)$ (since Y is cofinal in α) and $\mathrm{type}(Y) \leq \mathrm{cf}(\alpha)$ (since $\mathrm{type}(Y) = \mathrm{type}(\mathrm{Dom}(h))$ and $\mathrm{Dom}(h) \subset \mathrm{cf}(\alpha)$).
2. Suppose that $X \subset \alpha$ and $Y \subset \beta$ have the same type. By 1., let $X' \subset X$ be cofinal in α and have type $\mathrm{cf}(\alpha)$. By the order-isomorphism between X and Y, X' corresponds to some $Y' \subset Y$ that has the same type as X'. This shows $\mathrm{cf}(\beta) \leq \mathrm{type}(X') = \mathrm{cf}(\alpha)$. By symmetry, $\mathrm{cf}(\alpha) \leq \mathrm{cf}(\beta)$.
3. ω_α and α have cofinal subsets of the same type: $\{\omega_\xi \mid \xi < \alpha\}$, resp., α itself.

114. Prove Lemma 6.22.
Solution. 1. $\aleph_\alpha = \sum_{\xi < cf(\omega_\alpha)} p_\xi < \prod_\xi \aleph_\alpha = \aleph^{cf(\aleph_\alpha)}$.
2. If not, then: $2^{\aleph_\alpha} = \sum_{\xi<\omega_\alpha} p_\xi < \prod_\xi 2^{\aleph_\alpha} = (2^{\aleph_\alpha})^{\aleph_\alpha} = 2^{\aleph_\alpha}$. □

115. Suppose that α is a limit and that $\omega_\beta < cf(\omega_\alpha)$. Show that $\aleph_\alpha^{\aleph_\beta} = \sum_{\gamma<\alpha} \aleph_\gamma^{\aleph_\beta}$.
Solution. If $\gamma < \alpha$, then $\aleph_\gamma^{\aleph_\beta} \leq \aleph_\alpha^{\aleph_\beta}$, hence $\sum_{\gamma<\alpha} \aleph_\gamma^{\aleph_\beta} \leq \aleph_\alpha^{\aleph_\beta} \cdot \aleph_\alpha = \aleph_\alpha^{\aleph_\beta}$.

Also, $\aleph_\alpha^{\aleph_\beta} = |\omega_\alpha^{\omega_\beta}| = |\sum_{\gamma<\alpha} \omega_\gamma^{\omega_\beta}| \leq \sum_{\gamma<\alpha} \aleph_\gamma^{\aleph_\beta}$. □

116. (Hausdorff) Prove that $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}$.

Solution. If $\aleph_\beta \le \aleph_\alpha$, then: $\aleph_{\alpha+1}^{\aleph_\beta} = |\bigcup_{\xi<\omega_{\alpha+1}} \xi^{\omega_\beta}| \le \sum_{\xi<\omega_{\alpha+1}} \aleph_\alpha^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1} \le \aleph_{\alpha+1}^{\aleph_\beta}$.
If $\aleph_\beta \ge \aleph_{\alpha+1}$, then: $\aleph_{\alpha+1}^{\aleph_\beta} \le (2^{\aleph_\alpha})^{\aleph_\beta} = 2^{\aleph_\beta} \le \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1} \le \aleph_{\alpha+1}^{\aleph_\beta}$. □

117. Show that for all $n, m \in \omega$: $\aleph_n^{\aleph_m} = \aleph_n \cdot 2^{\aleph_m}$.
Solution. Induction w.r.t. n. For $n = 0$: $\aleph_0^{\aleph_m} \le (2^{\aleph_0})^{\aleph_m} = 2^{\aleph_m} \le \aleph_0 \cdot 2^{\aleph_m}$.
For $n + 1$, by Hausdorff's formula: $\aleph_{n+1}^{\aleph_m} = \aleph_{n+1} \cdot \aleph_n^{\aleph_m} \le \aleph_{n+1} \cdot \aleph_n \cdot 2^{\aleph_m} = \aleph_{n+1} \cdot 2^{\aleph_m}$. □

118. Show: $\aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1}$.
Solution. Every $g \in \omega_\omega^{\omega_1}$ can be split into ω parts $g_n := (\omega_1 \times \omega_n) \cap g$. This shows that $\aleph_\omega^{\aleph_1} \le \prod_n \aleph_n^{\aleph_1}$.
Alternatively: $\aleph_\omega^{\aleph_1} = (\sum_n \aleph_n)^{\aleph_1} \leqslant (\prod_n \aleph_n)^{\aleph_1} = \prod_n \aleph_n^{\aleph_1}$. The rest, using Exercise 117:
$\prod_n \aleph_n^{\aleph_1} = \prod_n (\aleph_n \cdot 2^{\aleph_1}) \le \prod_n (\aleph_\omega \cdot 2^{\aleph_1}) = (\aleph_\omega \cdot 2^{\aleph_1})^{\aleph_0} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1 \cdot \aleph_0} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1}$.

119. (Bukovsky; Hechler) Assume that $\aleph_\alpha$ is singular and that $\beta < \alpha$ exists such that $\beta \le \gamma < \alpha \Rightarrow 2^{\aleph_\gamma} = 2^{\aleph_\beta}$. Show that $2^{\aleph_\alpha} = 2^{\aleph_\beta}$.
Solution. Write $\aleph_\alpha = \sum_{\xi < cf(\omega_\alpha)} p_\xi$, where $p_\xi < \aleph_\alpha$. Since $cf(\aleph_\alpha) < \aleph_\alpha$, you can choose $\aleph_\gamma$ such that $cf(\aleph_\alpha), \aleph_\beta < \aleph_\gamma < \aleph_\alpha$. Then $cf(\aleph_\alpha) < 2^{\aleph_\gamma} = 2^{\aleph_\beta}$.
Thus: $2^{\aleph_\alpha} = 2^{\sum_\xi p_\xi} = \prod_\xi 2^{p_\xi} \le \prod_\xi 2^{\aleph_\beta} = (2^{\aleph_\beta})^{cf(\aleph_\alpha)} \le 2^{\aleph_\gamma} = 2^{\aleph_\beta}$. □

Chapter 7

134 Prove a few items of Lemma 7.12: give bounded formulas expressing the following:

1. $x = \emptyset$,
2. $x \subset y$,
3. $z = \{x\}$, $z = \{x, y\}$,
4. $z = x \cup y$,
   $z = x \cup \{y\} \leftrightarrow \forall u \in z(u \in x \ \vee \ u = y) \ \wedge \ \forall u \in x(u \in z) \ \wedge \ y \in z$.
5. $x = 0$, $x = 1$, $x = 2$, $x = 3, \ldots$,
   $x = n + 1 \leftrightarrow \exists y \in x(x = y \cup \{y\} \ \wedge \ y = n)$.
6. $x = \mathrm{V}_0$, $x = \mathrm{V}_1$, $x = \mathrm{V}_2$, $x = \mathrm{V}_3, \ldots$,
   $x = \mathrm{V}_{n+1} \leftrightarrow \exists y \in x(y = \mathrm{V}_n \ \wedge \ \forall z \in x(z \subset y) \ \wedge \ \emptyset \in x \ \wedge \ \forall z \in x \forall i \in y(z \cup \{i\} \in x)$.
7. x is 0, S-closed (i.e., $\emptyset \in x \ \wedge \ \forall y \in x(y \cup \{y\} \in x)$),
8. $x \in \mathrm{OR}$ (cf. Exercise 130),
9. α is a limit ordinal, α is a successor ordinal,
10. $x = \omega$, $x \in \omega$,
    $x \in \omega \leftrightarrow x = \emptyset \ \vee \ (\exists y \in x(x = y \cup \{y\}) \ \wedge \ \forall y \in x(y = \emptyset \ \vee \ \exists z \in x(y = z \cup \{z\})))$.
    Also: $x \in \omega$ iff $x \in \mathrm{OR}$, x is not a limit, and no $y \in x$ is a limit.
11. $y = \bigcup x$,
12. $z = (x, y)$ $(= \{\{x\}, \{x, y\}\})$,
13. p is an ordered pair,
14. R is a relation, xRy,
15. f is a function, $f(x) = y$, f is an injection, resp., surjection, resp., bijection,
16. $x = \mathrm{Dom}(f)$, $y = \mathrm{Ran}(f)$, $g = f|A$.

141 $\mathrm{L}_\alpha \subset \mathrm{V}_\alpha$.
Solution.
Induction w.r.t. α; use that $\mathrm{Def}(A) \subset \wp(A)$.

142 Every L_α is transitive. $\mathbf{L}$ is transitive.

Solution.
Induction w.r.t. α. The only non-trivial case is the successor step. Suppose that $x \in \mathrm{L}_{\alpha+1}$ and $y \in x$. Then $x \subset \mathrm{L}_\alpha$ and hence $y \in \mathrm{L}_\alpha$. By IH, $y \subset \mathrm{L}_\alpha$. Thus, $y = \{u \in \mathrm{L}_\alpha \mid (u \in y)^{\mathrm{L}_\alpha}\} \in \mathrm{L}_{\alpha+1}$.

143 If $\alpha < \beta$, then $\mathrm{L}_\alpha \in \mathrm{L}_\beta$, and, hence, $\mathrm{L}_\alpha \subset \mathrm{L}_\beta$.
Solution.
Use that $A \in \mathrm{Def}(A)$.

144 $\mathrm{OR} \cap \mathrm{L}_\alpha = \alpha \in \mathrm{L}_{\alpha+1} - \mathrm{L}_\alpha$; $\mathrm{OR} \subset \mathbf{L}$.
Solution.

159. Show that, in the reflection principle, $\{\alpha \mid A_\alpha \prec_\Sigma A\}$ is closed.
Solution.
(*Closed.*) Assume that $C_\Sigma = \{\xi \in \mathrm{OR} \mid A_\xi \prec_\Sigma A\}$ is unbounded in the limit ordinal α. We need to show that, for $\Phi \in \Sigma$, the equivalence $\Phi^{A_\alpha} \leftrightarrow \Phi^A$ holds on parameters from A_α.
Induction w.r.t. the nr of logical symbols in Φ. (*This* is the reason we're assuming that Σ is subformula closed. The proof is not unlike the one for the so-called *Elementary Chain Lemma* from model theory.) The only problem arises when Φ is a quantification; say, $\Phi = \exists z \Psi(x, y, z)$. Assume that $a, b \in A_\alpha$.
($\Rightarrow$) Suppose that $[\exists z \Psi(a, b, z)]^{A_\alpha}$. Say, $c \in A_\alpha$ is such that $[\Psi(a, b, c)]^{A_\alpha}$. By IH on Ψ, it follows that $[\Psi(a, b, c)]^A$. Hence, $[\exists z \Psi(a, b, z)]^A$.
($\Leftarrow$) Conversely, suppose that $[\exists z \Psi(a, b, z)]^A$. Choose $\xi < \alpha$ in C_Σ such that $a, b \in A_\xi$. Then since $\xi \in C_\Sigma$, we also have that $[\exists z \Psi(a, b, z)]^{A_\xi}$. Thus, $c \in A_\xi$ exists such that $[\Psi(a, b, c)]^{A_\xi}$, and it follows that $[\Psi(a, b, c)]^A$. By IH on Ψ, $[\Psi(a, b, c)]^{A_\alpha}$. Therefore, $[\exists z \Psi(a, b, z)]^{A_\alpha}$.
(*Unbounded.*) Induction w.r.t. the nr of formulas in Σ.
If $\Sigma = \emptyset$, then $C_\Sigma = \mathrm{OR}$ is club. Otherwise, choose $\Phi \in \Sigma$ with a maximal nr of logical symbols. Put $\Gamma = \Sigma - \{\Phi\}$. By IH, C_Γ is club. Distinguish as to the form of Φ.
(i) Φ is atomic. Then all formulas in Σ are atomic and $C_\Sigma = \mathrm{OR}$ is club.
(ii) $\Phi = \neg\Psi$. Then $\Psi \in \Gamma$, and $C_\Sigma = C_\Gamma$.
(iii) $\Phi = \Phi_1 \wedge \Phi_2$. Then $\Phi_1, \Phi_2 \in \Gamma$, and, again, $C_\Sigma = C_\Gamma$.
(iv) $\Phi = \exists x \Psi(x, y, z)$. Then $\Psi \in \Gamma$.
Suppose that $\alpha \in \mathrm{OR}$. We need to find $\beta \geq \alpha$ in C_Σ. Define a sequence $\alpha \leq \beta_0 \leq \beta_1 \leq \beta_2 \leq \cdots$ of ordinals in C_Γ such that

$$\beta_{n+1} = \bigcap\{\gamma \in C_\Gamma \mid \gamma \geq \beta_n \wedge \forall b, c \in A_{\beta_n}[\exists x \in A \Psi^A(x, b, c) \Rightarrow \exists x \in A_\gamma \Psi^A(x, b, c)]\}.$$

Note that β_{n+1} exists: choose, for every $b, c \in A_{\beta_n}$ s.t. $\exists x \in A \Psi^A(x, b, c)$ holds, an ordinal $\gamma_{b,c} \geq \beta_n$ for which $\exists x \in A_{\gamma_{b,c}} \Psi^A(x, b, c)$. Consider $\gamma = \bigcup\{\gamma_{b,c} \mid b, c \in A_{\beta_n}\}$ (Substitution and Sumset Axioms). Now γ satisfies the defining condition of (the class defining) β_{n+1}.
Now, let $\beta = \bigcup_n \beta_n$. Then $\beta \geq \alpha$. Also, $\beta \in C_\Gamma$, since C_Γ is closed.
To see that, in fact, $\beta \in C_\Sigma$, we finally check the equivalence $\Phi^{A_\beta} \leftrightarrow \Phi^A$ on parameters from A_β. Assume that $b, c \in A_\beta$.
($\Rightarrow$) If $\Phi^{A_\beta}(b, c)$, then $a \in A_\beta$ exists such that $\Psi^{A_\beta}(a, b, c)$. Since $\beta \in C_\Gamma$ and $\Psi \in \Gamma$, it follows that $\Psi^A(a, b, c)$. Thus, $\Phi^A(b, c)$.
($\Leftarrow$) Suppose that $\Phi^A(b, c)$; i.e., that $\exists x \in A \Psi^A(x, b, c)$. As $b, c \in A_\beta$, there exists n such that $b, c \in A_{\beta_n}$. By construction of β_{n+1}, $a \in A_{\beta_{n+1}}$ exists such that $\Psi^A(a, b, c)$. Since $\beta \in C_\Gamma$, it follows that $\Psi^{A_\beta}(a, b, c)$ and, hence, that $\Phi^{A_\beta}(b, c)$.

163. Show that $\mathbf{L}$ is absolute w.r.t. every transitive collection that contains all ordinals and satisfies sufficiently many ZF axioms.
Solution. Let Σ be the ZF-axioms needed to prove that $\forall \alpha \in \mathrm{OR} \exists y \mathcal{L}(\alpha, y)$ ($\mathcal{L}(\alpha, y)$ is a Σ_1-formula that, relative to ZF, amounts to $y = \mathrm{L}_\alpha$).
Claim: If K is transitive, $\mathrm{OR} \subset K$, and $(\forall \alpha \in \mathrm{OR} \exists y \mathcal{L}(\alpha, y))^K$, then $\mathbf{L}$ (which we take to be defined by the formula $\exists \alpha \exists y[\mathcal{L}(\alpha, y) \wedge x \in y]$) is absolute w.r.t. K.
Proof:

$\mathbf{L}^K \subset \mathbf{L}$: Suppose that $x \in \mathbf{L}^K$. That is: $\exists\alpha\exists y[\mathcal{L}(\alpha, y) \wedge x \in y]$ holds in K. Say, $\alpha \in \text{OR}$, $y \in K$, $x \in y$, $\mathcal{L}^K(\alpha, y)$. By upward persistence, $\mathcal{L}(\alpha, y)$ holds as well. Thus, $y = \text{L}_\alpha$, and $x \in \mathbf{L}$.
$\mathbf{L} \subset \mathbf{L}^K$: Suppose that $x \in \mathbf{L}$, say, $x \in \text{L}_\alpha$. By assumption on K, $y \in K$ exists such that $\mathcal{L}^K(\alpha, y)$. By persistence, $\mathcal{L}(\alpha, y)$, i.e.: $x \in \text{L}_\alpha = y$. Hence, $\exists\alpha\exists y[\mathcal{L}(\alpha, y) \wedge x \in y]$ holds in K.

166. Define $A^{<\omega} = \{f \mid f \text{ is a finite function s.t. } \text{Dom}(f) \subset \omega \wedge \text{Ran}(f) \subset A\}$. Show that The formula $X = A^{<\omega}$ is Σ_1^{ZF}.

Solution. $X = A^{<\omega}$ holds iff
$\emptyset \in X \wedge \forall g{\in}X\forall n{\in}(\omega - \text{Dom}(g))\forall a{\in}A[g \cup \{(n, a)\} \in X] \wedge$
$\wedge \forall g{\in}X\exists n{\in}\omega[g \text{ is a function } \wedge \text{Dom}(g) \subset n \wedge \text{Ran}(g) \subset A]$.

178. Suppose that $(A, <)$ is a wellordering and $f : A \to B$ a surjection. Define the relation $\prec$ on B by $x \prec y \equiv$ the $<$-first element of $f^{-1}(x)$ is $<$-smaller than the $<$-first element of $f^{-1}(y)$. Then $\prec$ wellorders B.
Solution. The correspondence: $x \mapsto \prec$-first element of $f^{-1}(x)$, embeds $(B, \prec)$ into $(A, <)$.

186. Show that the formula $x =_1 y$ (which is Σ_1^{ZF}) is not Π_1^{ZF} (unless ZF is inconsistent).
Solution. Assume that $x =_1 y$ is provably equivalent with the Π_1-formula $\Phi(x, y)$.
Reason in ZF. Choose two infinite sets a and b such that $a \neq_1 b$. Hence, $\neg\Phi(a, b)$ is true.
Reflection: choose A satisfying Extensionality such that $a, b \in A$ and $(\neg\Phi(a, b) \wedge a, b \text{ infinite})$ true in A.
Löwenheim-Skolem: choose a countable $B \subset A$ with $a, b \in B$ and $(\neg\Phi(a, b) \wedge a, b \text{ infinite})$ true in B.
Collapse B to a transitive C via an isomorphism h.
Then $(\neg\Phi(h(a), h(b)) \wedge h(a), h(b) \text{ infinite})$ is true in C, and since Φ is Π_1, $(\neg\Phi(h(a), h(b)) \wedge h(a), h(b) \text{ infinite})$ is true in $\mathbf{V}$. Thus, $h(a)$ and $h(b)$ are infinite sets of different cardinality.
However, $h(a), h(b) \subset C$. Thus, $h(a)$ and $h(b)$ are countably infinite, and hence have the same cardinality.

Chapter 8

245. ZFC is consistent with $2^\omega = \omega_1$, $2^{\omega_1} = \omega_3$ and $2^{\omega_2} = \omega_7$ simultaneously.
Proof. Assume GCH in the model you start from. Note that, by GCH: $\omega_7^{\omega_2} = \omega_7$. The trick is to work backwards. (If you start with $\text{Fn}_{\omega_1}(\omega_3, 2)$, you force $2^\omega = \omega_1$, $2^{\omega_1} = 2^{\omega_2} = \omega_3$. But, in order for $\text{Fn}_{\omega_2}(\omega_7, 2)$ to work, you need $2^{\omega_1} \leq \omega_2$.) Thus, *first* force with $\text{Fn}_{\omega_2}(\omega_7, 2)$. Then you get $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$ and $2^{\omega_2} = \omega_7$; moreover, $\omega_3^{\omega_1} = \omega_3$. *Next*, force with $\text{Fn}_{\omega_1}(\omega_3, 2)$.

Lemma 8.92 *A filter $G \subset P$ is $\mathbf{M}$-generic iff*

1. $1 \in G$,
2. $p \in G \wedge p \leq q \Rightarrow q \in G$,
3. $p, q \wedge G \Rightarrow \exists r(r \leq p, q)$,
4. *G intersects every maximal antichain in $\mathbf{M}$.*

Proof. $(\Rightarrow)$ If $A \subset P$ is a maximal antichain in $\mathbf{M}$, then $D = \{p \mid \exists q \in A(p \leq q)\}$ is dense.
$(\Leftarrow)$ 3: Suppose that $D \in \mathbf{M}$ is dense. Choose a maximal antichain $A \subset D$. Check that A is a maximal antichain. 2: The set $\{r \mid r \leq p, q \vee r \perp p \vee r \perp q\}$ is dense. □

240. Suppose that P is ccc and $G \subset P$ is $\mathbf{M}$-generic. Let $C \subset \omega_1$ be a club in $\mathbf{M}[G]$. Show that a club $C' \subset \omega_1$ in $\mathbf{M}$ exists such that $C' \subset C$.
Solution. Kunen gives the following hint. Pick $f \in \mathbf{M}[G]$ such that $\forall\alpha < \omega_1(\alpha < f(\alpha) \in C)$. (Thus, f witnesses unboundedness of C.) By ccc-ness, $F \in \mathbf{M}$ exists such that $\forall\alpha(f(\alpha) \in F(\alpha) \leqslant_1 \aleph_0)$.

Define $g(\alpha) = \bigcup F(\alpha)$. Note that $g \in \mathbf{M}$. Since $F(\alpha)$ is countable, $g(\alpha) < \omega_1$; furthermore, some $\xi \in C$ satisfies $\alpha < \xi \leqslant g(\alpha)$. It follows that $\bigcup_n g^n(\alpha) \in C$: pick $\xi_n \in C$ such that $g^n(\alpha) < \xi_n \leqslant g^{n+1}(\alpha)$; then $\bigcup_n \xi_n = \bigcup_n g^n(\alpha)$, and $\bigcup_n \xi_n \in C$ since C is closed.

Thus, $\{\bigcup_n g^n(\alpha) \mid \alpha < \omega_1\}$ is an unbounded subset of C in $\mathbf{M}$. Its closure in $\mathbf{M}$ is the subset C' required.

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