C.R.AIPS 4

Proceedings of *the Académie Internationale de Philosophie des Sciences*

Comptes Rendus de l'Académie Internationale de Philosophie des Sciences

Tome IV

The Relevance of Judgment for Philosophy of Science

Éditeur
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Axiomatic thinking and judgment

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Considering the *Relevance of Judgment for Philosophy of Science*, the motto of the AIPS conference 2021, we first ask what the term *judgment* means.

The Merriam-Webster dictionary offers the options given in Figure 1.

1. a : the process of forming an opinion or evaluation by discerning and comparing
   b : an opinion or estimate so formed
2. a : the capacity for judging : DISCERNMENT
   b : the exercise of this capacity
3. a : a formal utterance of an authoritative opinion
   b : an opinion so pronounced
4. a : a formal decision given by a court
   b (1) : an obligation (such as a debt) created by the decree of a court
     (2) : a certificate evidencing such a decree
5. a *capitalized* : the final judging of humankind by God
   b : a divine sentence or decision
   specifically : a calamity held to be sent by God
6. : a proposition stating something believed or asserted

Figure 1. From the website `merriam-webster.com`, accessed in March 2025.

In philosophy of science, we would like to exclude the *authoritative opinion* and the *divine sentence* as meanings, but the word *judgment* may have any of the other meanings mentioned. In jurisprudence, a judgment delivered by a court should be an *informed decision*. This means it must be justified, and every court has to provide a justification for its decision. In this context, the judgment is not to be confused with the justification; however, every such judgment requires a justification as a basis.

Even if "a proposition [is] stating something believed or asserted", we should have the right to ask: *Why?* The answer should, then, document both the process and the capacity for discernment.

In mathematics, we clearly understand how to answer Why-questions: by providing a proof. A proof establishes the truth of a proposition. This

The relevance of judgment for the philosophy of science, edited by Jure Zovko.
Comptes Rendus de l'Académie Internationale de Philosophie des Sciences 4 (2025).
R. Kahle, Axiomatic thinking and judgment, pp. 89–95.

fact itself is a judgment—this is symbolically expressed by what Frege called *judgment stroke* (*Urteilsstrich*).[1]

This is also the way that Per Martin-Löf introduced the notion of judgment in his logical framework:[2] "First of all, we have the notion of proposition. Second, we have the notion of truth of a proposition. Third, combining these two, we arrive at the notion of assertion or judgement.", illustrated by Figure 2 [ML87, p. 409].

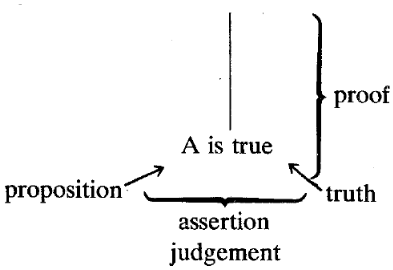

FIGURE 2. Illustration of "propostion", "proof", "truth" and "assertion or judgement" by Martin-Löf.

A proof requires an axiomatic setting.[3] And by use of a proper axiomatization of the fundamental terms of any science—a *framework of concepts* as Hilbert [Hil18] called it—, we can apply our notion of proof, first developed for mathematics only, to other scientific areas. Hilbert, in fact, advocated the application of the axiomatic method to every mature science: "I believe:

[1]Smith [Smi00] discusses at length an apparent "tension" in Frege's conception of the judgment stroke, which seems to maintain a psychological component in his logic. The paper provides a good review of the contemporary and later criticism of Frege concerning this tension and offers a solution by distinguishing "Frege's conception of logic and our own". We don't see any tension at all, if one distinguishes proofs as part of epistemology and truth as part of ontology. But it's worth noting that Frege himself did not make this distinction.

[2]Martin-Löf also gives an interesting historical answer to the question "How did 'judgement' come to be a term of logic?", see [ML11].

[3]Of course, a proof in the colloquial sense will essentially always be informal. But—that is the first lesson of Hilbert's programme—any informal proof should be translatable into a formal proof. This was explicitly acknowledged by Kurt Gödel [Gö95, p. 45]:

> The problem of giving a foundation for mathematics ... can be considered as falling into two parts. At first these methods of proof have to be reduced to a minimum number of axioms and primitive rules of inference ...; and then secondly a justification in some sense or other has to be sought for these axioms ....
>
> The first part of the problem has been solved in a perfectly satisfactory way, the solution consisting in the so-called "formalization" of mathematics ....

everything that can be object of scientific thinking in general, as soon as it is ripe for formation of a theory, runs into the axiomatic method and thereby indirectly to mathematics." This was not empty propaganda, as Richard Courant reminds us [Cou81]:

> At that time Max Born und Franck had come to Göttingen and Hilbert founded a special seminar with the physicists on the Structure of Matter. The term "structure of matter" really comes from Hilbert's seminar. . . . The seminar and Hilbert's inspiring interest played a very much greater role in this than the normal art-historical physicist of today knows, or even has the slightest idea about.

Ultimately, the concept of judgment transfers from logic to all other scientific areas which are axiomatized.

There is, however, a catch here. There might be everything correct with a judgment given by a proper derivation of a proposition from certain axioms. But how do we *judge* axioms?

Traditionally they were considered as *evident truths*. But, especially in mathematics, that would today amount to "an authoritative opinion"—as exemplified by the famous *parallel axiom*. The very possibility of non-Euclidean geometries taught mathematics that axioms cannot be taken for granted—despite the acknowledged authority of Euclid.

In fact, modern mathematics dispenses with the notion of absolute truth. Instead, it investigates different *structures*[4] using a unified methodology. This is evident for abstract algebra,[5] but also applies to arithmetic and geometry. For geometry, we already mentioned non-Euclidean geometry, but the case of spherical geometry is even more obvious. In arithmetic we apparently only consider the "one and only" structure of the natural numbers. One may ask, then, why we feel entitled to use usual number symbols on a clock face, when they are meant to represent elements of the cyclic group $\mathbb{Z}/12\mathbb{Z}$.

Hilbert was asked by Max Dehn where he got his axioms for geometry. He answered:[6] "*Studiere Pasch!*"—"Study Pasch!" And Pasch considered the justification of axioms to be an independent task, which can be dubbed *promathematics*, *propaedeutics*, or the *narthex of geometry*.[7]

This narthex is, indeed, an undeveloped field of modern logic.

For Pasch, when considering Geometry, *experience* is crucial in the narthex. Extending the axiomatic method to arbitrary areas of investigation,

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[4]Since there is not just one structure, Bernays [Ber50] can speak, quite correctly, of "*bezogene Existenz*" (related existence) with respect to existence claims in Mathematics.

[5]Bourbaki comments [Bou50, p. 225]: "It goes without saying that there is no longer any connection between this interpretation of the word 'axiom' [in an axiom system for groups] and its traditional meaning of 'evident truth'."

[6]See [Tam07, p. 62].

[7]See [Tam07, p. 80].

Hilbert liberalized the requirement and was to some extent only demanding consistency. Tamari [Tam07, p. 80] characterizes the difference neatly: The difference between Pasch and Hilbert is reduced to the fact that [Hilbert's] metamathematical method of proving consistency is a postcondition (*Nachtrag*) to the theory of geometry, whereas for Pasch it is a "precondition" ("*Vortrag*").

Pasch's justification is extrinsic (and, as one immediately notices, very much in line with Frege). But also Hilbert's approach is not a meaningless game of symbols. Axiom systems may be semantically motivated. Hilbert and Bernays [HB11, p. 2] make this clear: "Formal axiomatics requires contentual axiomatics as a necessary supplement. It is only the latter that provides us with some guidance for choosing the right formalism ...". And, additionally, the axiomatic method invites a certain form of idealization [HB11, p. 3]: "[I]n science we are predominantly—if not always—concerned with theories that do not reproduce the actual state of affairs completely, but whose significance consists in a *simplifying idealization* of the actual state of affairs."[8]

It was only later that Bourbaki turned the game around. Mathematics does not need to follow semantic specifications; rather, it provides alternative structures—here to be understood as given in an axiomatic way—from which one can choose [Bou50, p. 231]:

> From the axiomatic point of view, mathematics appears thus as a storehouse of abstract forms—the mathematical structures; and it so happens—without our knowledge why—that certain aspects of empirical reality fit themselves into these forms, as if through a kind of preadaptation. Of course, it can not be denied that most of these forms had originally a very definite intuitive content; but, it is exactly by deliberately throwing out this content, that it has been possible to give these forms all the power which they were capable of displaying and to prepare them for new interpretations and for the development of their full power.

When Mathematics produces such *abstract forms* for Bourbaki's storehouse, it can dispense with specific judgments concerning the choice of the axioms—although they may be, and probably will be, motivated by the original intuitive content, conceded by Bourbaki.

Interestingly, the philosopher Jürgen Habermas saw here—most likely without any knowledge of Bourbaki or even the axiomatic method—a new understanding of science, which went against his hope for societal engagement [Hab57, p. 65]:

[8] Georg Kreisel repeatedly criticised Hilbertian proof theory for studying only the structure of given axiomatic systems without addressing the question of where the axioms come from. He apparently deliberately ignored the opening paragraphs of Hilbert and Bernays's *Grundlagen der Mathematik*.

> Applicable science is neutral toward a social practice that now decides how to apply it without scientific guidance. It pays for its practicability by alienating itself from living, purposeful practice, becoming a "pure theory".

The axiomatic method seems, indeed, to support this estrangement of science from social practice. And Bourbaki even attempts to absolve itself of any responsibility for applications [Bou49, p. 2]:

> Why do such applications ever succeed? Why is a certain amount of logical reasoning occasionally helpful in practical life? Why have some of the most intricate theories in mathematics become an indispensable tool to the modern physicist, to the engineer, and to the manufacturer of atom-bombs? Fortunately for us, the mathematician does not feel called upon to answer such questions, nor should be held responsible for such use or misuse of his work.

But we may identify here a new quest for judgment: how is the choice of one abstract form over another justified when, for instance, a physicist picks it up from the storehouse?

We can refer back to Kant here. He stressed the *purposefulness* of axioms. And purposefulness, of course, applies also to abstract forms one would like to use. As a matter of fact, Euclidean geometry does not serve the purpose if it is used to investigate movements of light points in the night sky, as well as for the geometry of the surface of the Earth. That justifies (the introduction and) the use of spherical geometry.

In his *Kritik der Urteilskraft*, Kant had linked the *purposefulness*[9] (as a *principle a priori*) with *judgment* (as a *cognitive faculty*) but in Aesthetics for applications in art. We actually see a quite similar relationship between purposefulness and judgment in Philosophy of Science: an abstract form is judged with respect to its purpose.

There is another Kantian observation which supports our view. In the *Kritik der reinen Vernunft* he writes [Kan98, p. 241 (A125)]:

> Thus we ourselves bring into the appearances that order and regularity in them that we call nature, and moreover we would not be able to find it there if we, or the nature of our mind, had not originally put it there.

The bold interpretation is this: the *framework of concepts* is imposed on nature by us—and by *judgment*.

Acknowledgement. Research supported by national funds through the FCT – Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 and UIDP/00297/2020 (Center for Mathematics and Applications) and by the Udo Keller Foundation.

[9] J. H. Bernard translates *Zweckmäßigkeit* as *purposiveness*, [Kan51, p. 34].

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