C.R.AIPS 2

Proceedings of *the Académie Internationale de Philosophie des Sciences*

Comptes Rendus de l'Académie Internationale de Philosophie des Sciences

Tome II

Justification, Creativity, and Discoverability in Science

Éditeur

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Justification, creativity, and discoverability in mathematics: The example of predicativity

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Abstract. The main thesis put forward in this paper is that the norm of predicative definability is a relative concept which therefore greatly affects its philosophical relevance. Predicative definitions even risk being a miss if they are not considered as an indication to abandon classical continuous analysis.

After an overview of the developmental stages of the mathematical specification of the intuitive concept of predicativity as per Russell and Poincaré, in section three I discuss different historical approaches to justify the principle of complete induction, and ask whether it is possible to avoid its impredicativity. If predicativity is considered from an extensional perspective, complete induction would possess an irreducible impredicative character even though it is not treated as an explicit definition but as an inductive definition. By contrast, if predicativity is considered from an intensional perspective, a purely operational and predicative justification of complete induction using operative imagination (Lorenzen) would be possible.

Mathematical practice depends on epistemic norms, which are themselves influenced by mathematical developments which, in turn, influence the standards for ontological, epistemological and semantical questions.

1 Introduction

There is no standard account of the condition to be fulfilled in order to justify the general concept of mathematical definition, nor how the distinction between suspicious procedures should be drawn. In efforts to justify the epistemically significant nature of mathematical definition, different criteria have been proposed, some of them depending on the framework of object-realism, formalism, and intuitionism. I will confine myself to the criterion of predicativity, which was advanced as philosophically motivated by proponents of a variant of constructivism.

My aim is to show that even with respect to predicativity there is only a vague context-dependent boundary between evident and suspect definition: the mathematical implementation of the normative philosophical criterion remains vague both in terms of its technical scope and its philosophical stringency.

In the second section, I take Russell's and Poincaré's perspective as a starting point in order to present the technical specification of the concept of predicativity in terms of its historical development. In the third section, two attempts to justify the induction principle by Hilbert/Bernays and Lorenzen are then examined as a case study with regard to its predictive character.

Justification, Creativity, and Discoverability in Science, edited by Lorenzo Magnani.
Comptes Rendus de l'Académie Internationale de Philosophie des Sciences 2 (2024).
G. Heinzmann, Justification, creativity, and discoverability in mathematics, pp. 115–123.

2 Predicative definability: from Russell-Poincaré to Weyl, Lorenzen and Wang

For a detailed version of this section, see Heinzmann & van Atten (2022, 223–256).

In the wake of the discovery of the famous Russell paradox, Bertrand Russell and Henri Poincaré set out to answer the question 'Which propositional functions define sets in a non-circular way?' In a second step, the creativity of the search centered on the more general question 'Which mathematical sets (respectively concepts) are non-circularly definable?'.

In his *Principles of Mathematics*, Russell noted:

> "Having dropped the former [the axiom of comprehension], the question arises: Which propositional functions define classes which are single terms as well as many, and which do not? And with this question, our real difficulties begin" (Russell 1903, 103).

In 1907, he introduced a new terminology to solve the problem:

> "Norms[1] (containing one variable) which do not define classes I propose to call *non-predicative*; those which do define classes I shall call *predicative*" (Russell 1907, 34).

In his discussion of Russell (1907), Poincaré (1906) confirmed the non-predicative direct or indirect definitions (existence postulates = propositions) as circular: he labeled a definition 'predicative' if, in the *definiens*, the *definiendum* does not occur, and no reference is made to it: Otherwise, it was considered non-predicative.

For Poincaré, the circularity based on a non-predicative definition in Russell's antinomy is the sign of the Cantorian's "realistic" error, i.e., considering a totality as a *datum* independent of the construction of its individuals.[2] This is trivially the case for actual infinite sets that are to refuse.

Poincaré then gave two definitions of predicativity:[3]

$P(1)$ leads to the idea of a predicative definition which imposes a limit on the unrestricted quantification over sets which are available to us in a "constructive" sense: *it places a constructive restriction at the object level.*

$P(2)$ indicates restrictive conditions imposed on the quantification without an explicit restriction of the domain: for a classification to be predicative, it is sufficient that the quantification over an indefinite domain,[4]

[1] Russell calls propositional functions "norms" here.

[2] The definition of E in $\forall X(X \in E \leftrightarrow X \notin X)$ is non-predicative, since the *definiendum* E is itself a possible totality of the variation domain of a universal quantifier.

[3] Cf. Heinzmann (1985, chap. IV).

[4] A domain is indefinite, if we can add an element to it which cannot be expressed by the means of previously fixed definitions.

> on which the *definiendum* depends, does not change the already determined classification of its elements: *it places a constructive limitation at the level of description.*

There remains the problem of the extensional equivalence between $P(1)$ and $P(2)$. Will they exclude the same definitions? As we will see, new light was shed on this question only in the 1960s.

In his seminal book *Das Kontinuum* (1918), Hermann Weyl held that Russell's way out of his antinomy, the "type theory", made mathematics unworkable. Weyl is in all probability the initial proponent of the predicative definition of real numbers. It is based on an *iterative* formation of ideal objects with respect to the domain of natural numbers, equipped with its operations and presupposed—contrary to Poincaré—in a Platonist way. Weyl rejects the set-theoretic reconstruction of natural numbers, as our grasp of the basic concepts of set theory depends on a prior intuition of natural numbers.

In his system, he called a formula arithmetical if it does not contain bound *set* variables. An arithmetical formula defines a property that refers only to the totality of natural numbers but does not refer to the totality of sets of natural numbers. This leads to a system, called ACA, containing an Arithmetical Comprehension Axiom:

$$\exists X \forall x [x \in X \leftrightarrow \varphi(x)]$$

for each arithmetical φ, where the variable X is not in φ.

The system ACA is a conservative extension of Peano Arithmetic, even though it employs second-order concepts. This enables Weyl to recover a substantial amount of Analysis. Nevertheless, predicative mathematics is restricted to arithmetically definable sets. What matters is that the usual *Least Upper Bound Axiom* (LUB), which states that every set of reals which is bounded above has a least upper bound, is not valid because it involves an impredicative definition (Weyl 1918, 77).

So, the question arises: Can the technical implementation of predicativity lead to a less constrained revision of mathematics?

This brings us to the work of Paul Lorenzen. According to him, the requirement to accept only "definite" propositions is a methodological boundary to fully grasping the aspect of mathematics which can be regarded as "stable" or "safe". He defines 'definite' as follows:

(1) Any proposition decidable by schematic operations is called "definite".

(2) If a definite proof or refutation concept is fixed for a proposition, then the proposition itself is also definite, more precisely proof-definite or refutation-definite" (Lorenzen 1955, 5–6).

He notes that

(i) Non-predicative *concept formation* (sic) is indefinite and therefore excluded from operative mathematics.

(ii) Quantifiers are permissible provided that the formulas in the quantification domain are definite.

(iii) The natural numbers and Peano axioms, together with the definitions of addition, multiplication, and exponentiation, can be constructed as definite.

However, Lorenzen's real numbers are not a model of an ordered and complete Archimedean field.

Hao Wang further developed the idea of predicative mathematics as a justified part of mathematics with explicit reference to Lorenzen. Both, Wang and Lorenzen, aimed at transfinitely iterating the construction of definable sets in their systems of ramified analysis. Nevertheless, unlike Lorenzen, Wang accepted classical logic.

Wang's idea was to start from a multi-layered constructive set-theoretical ordered hierarchy and to ask whether one can then provide a more accurate characterization of predicativity (Wang, 1964, 578). His most important contribution was the discussion of the relationship between predicativity and ordinals. He related predicative defined sets to constructive ordinals by establishing a hierarchy as the union of all systems Σ_α, where α is a constructive ordinal.

This hierarchy does not lead beyond recursive ordinals! (Spector 1955). Kreisel subsequently formulated the thesis that all predicatively definable sets belong to Σ_{ω_1}, where ω_1 is the upper bound of recursive ordinals. Wang's hierarchy is a prime example of formalizing the intuitive idea of predicativity expressed by Poincaré in his first definition $P(1)$: it limits the quantification on already constructed sets, which comes down to a restriction in terms of construction.

Assuming the totality of natural numbers,[5] and presupposing a 2^{nd} order language enabling quantifications over sets of natural numbers, on the basis of preliminary work by Georg Kreisel (1960), Solomon Feferman (1964) proposed two definitions to predicativity: one which amounts to a constructive restriction at the object level and corresponds to $P(1)$, and one which comprises an extension of the domain of their 2^{nd} order quantifiers and corresponds to $P(2)$. He then shows that, in both cases, the predicatively definable sets are the same. In this way, Poincaré's intuition is confirmed at a higher level.

[5]The question of their predicativity is not addressed.

Finally, in 1959 Stephen Cole Kleene proved that Σ_{ω_1} exactly coincides with what is referred to as the class of hyperarithmetical sets Δ^1_1. Does Δ^1_1 express the central idea of predicativity in a clear way? No, because we can conclude from the falsity of a proposition, if relativized to Δ^1_1, to its non-predicativity, but not from the validity of a theorem in hyperarithmetical analysis to its predicativity. Predicative analysis seems to be somewhere between arithmetic and hyperarithmetical Analysis.[6]

There is yet another difficulty. The class of hyperarithmetic sets only specifies for the non-predicativiste what the predictive universe should be: Indeed, for a recursive ordinal it must be proven that not only its definition is predicative, but also that its ordering is predicatively recognized as being a well order by using principles of reasoning that had already been shown to be predicatively acceptable at a previous stage (Kreisel 1960, 387). This is why Kreisel, Feferman, and Kurt Schütte introduced the concept of *predicative provability*. I am unable to discuss this concept here.[7]

3 Is it possible to avoid the impredicativity of the induction principle in mathematics?

According to Hilbert's and Bernays' *Grundlagen der Mathematik* (1934, §2), the method of proof of complete induction is obtained from iteration by a further step involving "experiments in the mind". How should we thus imagine this further step of an experiment in the mind? In fact, it is obtained by adding to the iteration schema

$$\begin{aligned} &(a)\quad \Rightarrow \mathsf{I}\ [\text{"we can construct } \mathsf{I}\text{"}] \\ &(b)\quad n \Rightarrow n\mathsf{I}\ [\text{"if we have } n\text{, we can construct } n\mathsf{I}\text{"}] \end{aligned} \qquad (S)$$

the final clause

(c) We can obtain all numerals by application of the scheme S.

This clause (c) does not follow analytically from clauses (a) and (b) of S.

[6]Starting from arithmetical sets or relations, we obtain Σ^1_1 and Π^1_1 by an existential (respectively universal) quantification of the second order. Δ^1_1 designates the intersection of these two classes.

[7]In the 1960s, independently of each other Kurt Schütte and Solomon Feferman discovered that a certain ordinal limit Γ_0 plays for the so-called predicative Analysis a role analogous to the one played by ε_0 for arithmetic: for each well order of type $\alpha < \Gamma_0$ there is a proof that the order in question is a well order and that it uses exclusively order types $< \alpha$, i.e., it is the smallest ordinal whose well order is no longer *predictively* provable.

Modulo some further abstractions, we have thus returned to what is called a recursive definition of natural integers:

(a) $N(\mathsf{I})$ ["I is a number"]

(b) $N(n) \Rightarrow N(n\mathsf{I})$ ["if n is a number, its successor is also a number"] (S′)

(c) We can obtain all numbers by application of (a) and (b).

Now, the justification of the *schema* of complete induction

$$[E(0) \wedge \forall x(N(x) \wedge E(x) \rightarrow E(x'))] \rightarrow \forall x(N(x) \rightarrow E(x)) \qquad \text{(T)}$$

where T applies to any property E is correlative to S′, which means that the final clause c) cannot be deduced from the clauses a) and b): an application of T is needed. In other words, T implies S′, and S′ implies T. However, the final clause expressing that we can obtain all numbers by application of a) and b) amounts to taking N to be 'minimal', but N should be first and foremost defined by (a), (b), and (c)!

As a result, even without using an explicit second-order definition for induction, this inductive definition is impredicative. Other examples of the attempt at predicative reduction of induction are discussed in Parsons (1992).

Expressed in a terminology I introduced in a recent article on thought experiments (Heinzmann 2022), numerals constructed using the rule S constitute the *experimental realm* belonging to the general Kantian scheme G (universal) of string repetition. S is imaginatively (by means of a very far-reaching intuition) related to S′, which is symbolically interdependent with T.

Hilbert/Bernays 'sees' in an apocryphal (intuitive) way the relation between the iteration rule S and the inductive definition S′ without being able to deduce S′ by logical means: *it is a genuine thought experiment* that could be confirmed by examples of real experiments (=calculations).

In mathematical thought experiments, we take recourse—based on mathematical experiments—to a modal deviation using further semiotic means. *Epistemic intuition* provides us with access to these deviations as 'genuine possibilities' of mathematical inferences as opposed to 'pure fictions'. This accessibility is the justification for the validity claim concerning mathematical thought experiments (Heinzmann, 2022).

Now, Lorenzen argues that in his operative approach it is possible to obtain the Peano axioms by avoiding the impredicativity of an inductive definition of induction without recourse to an apocryphal intuition. In fact, for him, the induction principle is a predicative (= definite) *meta-rule* (independent of the language level of A) of the form

$$A(I);\ A(m) \to A(m\mathsf{I}) \Rightarrow A(n),\ n \text{ arbitrary}$$

that constitutes an operative interpretation of the classical induction principle. In fact, the formula in the conclusion is definite: its range consists exclusively of numerical signs constructed according the rules

$$\Rightarrow \mathsf{I}$$
$$m \Rightarrow m\mathsf{I}$$

and all numerical signs are the result of such a construction.

The acceptance of the predicative induction rule firstly points to a shift in meaning concerning the term used: impredicativity no longer refers to the definition of sets, but of concepts! Therefore, Parsons (1992, 152–154) is correct in his assertion that Lorenzen's concept of predicativity is novel and that is not so firmly entrenched as the classical interpretation of Poincaré's or Russell's definition of predicativity. However, it is not incompatible with Poincaré's definition: the circle lies in the fact that one speaks of sets that could not be extensions of *predicates* antecedently understood. In the same way, for Lorenzen, sets always should be in the range of predicates antecedently understood as definite: we should respect the conceptual order that places the understanding of the predicate before the apprehension of its extension as an object. He eschews set theoretic realism and considers the inductive rules as giving us an understanding of the predicate 'natural number'. The understanding of the predicate occurs prior to the insight that the set exists (Parson 1992, 254).

Lorenzen's formulation of induction can thus be read as a special case of an application of a constructive version of the ω-rule, which states that given a recursive function f such that for every natural number n the value $f(n)$ is the Gödel number of a proof of $A(n)$, one may proceed to: *for all* n, $A(n)$.

One thus obtains a complete semi-formalism of arithmetic without (probably) using actual infinity. However, the scope of 'extensional' and 'intensional' predicativity is now different: Kreisel (1959) shows that the Cantor-Bendixson theorem in Analysis (every closed set is the union of a perfect set and of a countable set) involves impredicative definitions, given that it does not hold in Δ^1_1, whereas Lorenzen and Myhill (1959) implicitly ascribed predicativity to the theorem as a consequence of their use of generalized inductive definitions.

4 Conclusion

The uncertainty of the predicativity of complete induction remains open. There are two kinds of predicativism settled between realism in extension (Platonism) and intuitionism:

I. Extensional predicativism (Poincaré, Weyl),

II. Intensional predicativism (Poincaré, Lorenzen).

In the first case, predicative definability leads to thought experiments—where we resort to a modal deviation of a logical inference using further semiotic means, in the second case to a semi-formalism. The difficult question to decide is this: What is preferable for understanding complete induction, a thought experiment (Hilbert) respectively pure intuition (Poincaré) on the object level, or operative imagination on the 'practical' meta-level? If one is convinced that the first important thing in mathematics is not proof but conceptual construction, Lorenzen's predicative definiteness implying a revisionist position gives predicative insights into classical non-predicative constructions.

Is predicativity a miss or perhaps a hint that one should abandon full formalisms, or even continuous Analysis, or should one return to "pre-Cartesian" geometric-topological intuition, as suggested by Poincaré and Bernays (1979, 13–14)? In philosophical terms, continuity cannot be adequately described by a full formalism without being a set theoretical realist. Nonetheless, difficulties in such a realism were precisely the motivation for Poincaré to invent predicative definability . . . and now predicativity requires thought experiments or semi-formal systems! We continue to remain in a state of vagueness.

Acknowledgements. I thank Paolo Mancosu for an exchange on predicativity which motivated this presentation. Lectures in Irvine (April 2023) and Buenos Aires (DLMPST, July 2023) were based on variants of this paper. I would like to thank the Service "traduction" de la MSH Lorraine for correcting the English language.

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